

The second-order contingent derivative of generalized perturbation maps

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ABSTRACT

In the paper, we give some remarks on [1]. Then, we modify main results concerning the sum rule of second-order contingent derivatives for set-valued maps and its application to the sensitivity analysis of

generalized perturbation maps. The obtained results are new and better than those in [1]. Some examples are proposed to illustrate our results.

Key words: second-order proto-differentiability, second-order semi-differentiability, metric subregularity, generalized perturbation map, set-valued map

INTRODUCTION

In [1], the second-order proto-differentiability and second-order semi-differentiability for set-valued maps were firstly discussed and applied to sum rules of two set-valued maps. Then, the authors established second-order sensitivity analysis of generalized perturbation maps as an application of sum rules. The semi-differentiability plays an essential role in all main results in [1].

In the paper, we give some remarks on the Proposition 2 and Theorem 1 in [1]. On the other hand, a new result is proposed to avoid the semi-differentiability by using a weaker hypothesis of the proto-differentiability.

The layout of this paper is as follows. Section 2 is devoted to several concepts needed in the sequel. Our main remarks and modified results are given in Section 3.

PRELIMINARIES

Throughout the paper, let X and Y be normed spaces. For a set-valued map $F : X \rightarrow 2^Y$, the domain, image, and graph of F are defined, respectively (resp for short), by

$$\begin{aligned} \text{dom}(F) &:= \{x \in X \mid F(x) \neq \emptyset\}, \quad \text{im}(F) := \{y \in Y \mid y \in F(X)\}, \\ \text{gr}(F) &:= \{(x, y) \in X \times Y \mid y \in F(x)\}. \end{aligned}$$

Definition 2.1 ([2, 3]). Let $S \subseteq X$, $x \in \text{cl}(S)$ and $w \in X$, where $\text{cl}(S)$ denotes the closure of S .

(i) The contingent cone and the adjacent cone of S at x are defined by, resp,

$$\begin{aligned} T(S, x) &:= \{u \in X \mid \exists t_n \rightarrow 0^+, \exists u_n \rightarrow u, x + t_n u_n \in S\}, \\ T^b(S, x) &:= \{u \in X \mid \forall t_n \rightarrow 0^+, \exists u_n \rightarrow u, x + t_n u_n \in S\}. \end{aligned}$$

(ii) The second-order contingent cone and the second-order adjacent cone of S at x in the direction w are defined by, resp,

$$T^2(S, x, w) := \left\{ u \in X \mid \exists t_n \rightarrow 0^+, \exists u_n \rightarrow u, x + t_n w + \frac{t_n^2}{2} u_n \in S \right\},$$

$$T^{b(2)}(S, x, w) := \left\{ u \in X \mid \forall t_n \rightarrow 0^+, \exists u_n \rightarrow u, x + t_n w + \frac{t_n^2}{2} u_n \in S \right\}.$$

Remark 2.1. From the Observation 1 in [4], we obtain the equivalent formulae of Definition 2.1(ii) as follows

$$T^2(S, x, w) := \left\{ u \in X \mid \begin{array}{l} \exists \alpha_n, \beta_n > 0 : \alpha_n \rightarrow +\infty, \beta_n \rightarrow +\infty, \frac{\beta_n}{\alpha_n} \rightarrow 2, \\ \exists x_n \in S : \alpha_n(x_n - x) \rightarrow w, \beta_n(\alpha_n(x_n - x) - w) \rightarrow u \end{array} \right\},$$

$$T^{2(b)}(S, x, w) := \left\{ u \in X \mid \begin{array}{l} \forall \alpha_n, \beta_n > 0 : \alpha_n \rightarrow +\infty, \beta_n \rightarrow +\infty, \frac{\beta_n}{\alpha_n} \rightarrow 2, \\ \exists x_n \in S : \alpha_n(x_n - x) \rightarrow w, \beta_n(\alpha_n(x_n - x) - w) \rightarrow u \end{array} \right\}.$$

Definition 2.2 ([2, 3]). Let $F : X \rightarrow 2^Y$, $(x, y) \in gr(F)$ and $(w, r) \in X \times Y$.

(i) The contingent derivative (the adjacent derivative) of F at (x, y) is a set-valued map $DF(x, y) : X \rightarrow 2^Y$ ($D^b F(x, y) : X \rightarrow 2^Y$, resp) such that

$$gr(DF(x, y)) := T(gr(F), (x, y))$$

$$(gr(D^b F(x, y))) := T^b(gr(F), (x, y)), \text{ resp}).$$

(ii) The second-order contingent derivative (the second-order adjacent derivative) of F at (x, y) in the direction (w, r) is a set-valued map $D^2 F(x, y, w, r) : X \rightarrow 2^Y$ ($D^{2(b)} F(x, y, w, r) : X \rightarrow 2^Y$, resp) such that

$$gr(D^2 F(x, y, w, r)) := T^2(gr(F), (x, y), (w, r))$$

$$(gr(D^{2(b)} F(x, y, w, r))) := T^{2(b)}(gr(F), (x, y), (w, r)), \text{ resp}).$$

Remark 2.2. From Definition 2.1 and Remark 2.1 is follous

$$D^2 F(x, y, w, r)(u) = \left\{ v \in Y \mid \exists t_n \rightarrow 0^+, \exists u_n \rightarrow u, \exists v_n \rightarrow v, y + t_n r + \frac{t_n^2}{2} v_n \in F \left(x + t_n w + \frac{t_n^2}{2} u_n \right) \right\}$$

$$= \left\{ v \in Y \mid \begin{array}{l} \exists \alpha_n, \beta_n > 0 : \alpha_n \rightarrow +\infty, \beta_n \rightarrow +\infty, \frac{\beta_n}{\alpha_n} \rightarrow 2, \\ \exists (x_n, y_n) \in gr(F) : \alpha_n(x_n - x) \rightarrow w, \beta_n(\alpha_n(x_n - x) - w) \rightarrow u, \\ \alpha_n(y_n - y) \rightarrow r, \beta_n(\alpha_n(y_n - y) - r) \rightarrow v \end{array} \right\},$$

$$D^{b(2)} F(x, y, w, r)(u) = \left\{ v \in Y \mid \forall t_n \rightarrow 0^+, \exists u_n \rightarrow u, \exists v_n \rightarrow v, y + t_n r + \frac{t_n^2}{2} v_n \in F \left(x + t_n w + \frac{t_n^2}{2} u_n \right) \right\}$$

$$= \left\{ v \in Y \left| \begin{array}{l} \forall \alpha_n, \beta_n > 0 : \alpha_n \rightarrow +\infty, \beta_n \rightarrow +\infty, \frac{\beta_n}{\alpha_n} \rightarrow 2, \\ \exists (x_n, y_n) \in gr(F) : \alpha_n(x_n - x) \rightarrow w, \beta_n(\alpha_n(x_n - x) - w) \rightarrow u, \\ \alpha_n(y_n - y) \rightarrow r, \beta_n(\alpha_n(y_n - y) - r) \rightarrow v \end{array} \right. \right\}.$$

Definition 2.3 ([5]). Let $F : X \rightarrow 2^Y$, $(x, y) \in gr(F)$ and $(w, r) \in X \times Y$. The second-order lower Dini derivative of F at (x, y) in direction (w, r) is a set-valued map $D_l^2 F(x, y, w, r) : X \rightarrow 2^Y$ that is defined by

$$D_l^2 F(x, y, w, r)(u) = \left\{ v \in Y \mid \forall t_n \rightarrow 0^+, \forall u_n \rightarrow u, \exists v_n \rightarrow v, y + t_n r + \frac{t_n^2}{2} v_n \in F \left(x + t_n w + \frac{t_n^2}{2} u_n \right) \right\}.$$

Remark 2.3. (i) By the proof similar to that of Observation 1 in [4], we get

$$D_l^2 F(x, y, w, r)(u) = \left\{ v \in Y \left| \begin{array}{l} \forall \alpha_n, \beta_n > 0 : \alpha_n \rightarrow +\infty, \beta_n \rightarrow +\infty, \frac{\beta_n}{\alpha_n} \rightarrow 2, \\ \forall x_n \in dom(F) : \alpha_n(x_n - x) \rightarrow w, \beta_n(\alpha_n(x_n - x) - w) \rightarrow u, \\ \exists y_n \in F(x_n) : \alpha_n(y_n - y) \rightarrow r, \beta_n(\alpha_n(y_n - y) - r) \rightarrow v \end{array} \right. \right\}.$$

(ii) It is obvious to see that

$$D_l^2 F(x, y, w, r)(u) \subseteq D^{b(2)} F(x, y, w, r)(u) \subseteq D^2 F(x, y, w, r)(u).$$

Definition 2.4 ([6, 7]). Let $F : X \rightarrow 2^Y$, $(x, y) \in gr(F)$ and $(w, r) \in X \times Y$.

(i) The map F is said to be second-order proto-differentiable at x relative to y in the direction (w, r) if $D^2 F(x, y, w, r) \equiv D^{b(2)} F(x, y, w, r)$.

(ii) The map F is said to be second-order semi-differentiable at x relative to y in the direction (w, r) if $D^2 F(x, y, w, r) \equiv D_l^2 F(x, y, w, r)$.

It is easy to see that if F is second-order semi-differentiable then F is second-order proto-differentiable.

Definition 2.5 ([1]). Let $F : X \rightarrow 2^Y$, $(x, y) \in gr(F)$ and $(w, r) \in X \times Y$. The map F is said to be second-

order lower semi-differentiable at (x, y) in the direction (w, r) if for any $\alpha_n, \beta_n > 0$, $x_n \in dom(F)$ with $\alpha_n \rightarrow +\infty, \beta_n \rightarrow +\infty, \frac{\beta_n}{\alpha_n} \rightarrow 2$, $\alpha_n(x_n - x) \rightarrow w$ and $\beta_n(\alpha_n(x_n - x) - w) \rightarrow u$ for some $u \in X$, there exists a subsequence $y_n \in F(x_n)$ such that $\alpha_n(y_n - y) \rightarrow r$ and $\beta_n(\alpha_n(y_n - y) - r)$ is convergent.

Remark 2.4. By Remark 2.3(i), if F is second-order semi-differentiable then F is second-order lower semi-differentiable. This assertion can be also implied immediately by Proposition 1 in [1].

Recall that a set-valued map $F : X \rightarrow 2^Y$ is called to be metric regular at $(x, y) \in gr(F)$ if there are $\mu, r > 0$ such that for all $u \in B_X(x, r), v \in B_Y(y, r)$,

$$d(u, F^{-1}(v)) \leq \mu d(v, F(u)), \tag{1}$$

where $B_x(x, r)$ denotes the open ball in X centered at x with radius r .

By fixing $v = y$ in (1), we get a weaker definition named by metric subregularity. The metric (sub)regularity plays an important role in variational analysis and has been applied to many topics of optimization, see [2, 8-10] and the references therein.

Inspired of the above definition, we propose the following concept.

Definition 2.6. Let $F : X \rightarrow 2^Y$, $(x, y) \in gr(F)$ and $S \subseteq X$. The map F is said to be metric subregularity at $(x, y) \in gr(F)$ with respect to S if there are $\mu, r > 0$ such that for all $u \in B_x(x, r) \cap S, v \in B_y(y, r)$,

$$d(u, F^{-1}(y) \cap S) \leq \mu d(y, F(u)).$$

MAIN RESULTS

Firstly, we recall a main proposition in [1] as follows.

Proposition 3.1 (Proposition 2 in [1]). *Suppose that $\bar{G} : X \rightarrow 2^Y$ is second-order proto-differentiable at x relative to $\bar{y} \in \bar{G}(x)$ in the direction $(w, \bar{r}) \in X \times Y$, $G : X \rightarrow 2^Y$ is second-order semi-differentiable at x relative to $y \in G(x)$ in the direction $(w, \hat{r}) \in X \times Y$ and G is second-order lower semi-differentiable at (x, y) in the direction (w, \hat{r}) . Then, the set-valued map $G : X \rightarrow 2^Y$ defined by $G := \bar{G} + G$ is second-order proto-differentiable at x relative to $y := \bar{y} + y$ in the direction (w, r) , where $r := \bar{r} + \hat{r}$, and for all $u \in X$,*

$$D^2G(x, y, w, r)(u) = D^2\bar{G}(x, \bar{y}, w, \bar{r})(u) + D^2G(x, y, w, \hat{r})(u)$$

Remark 3.1. (i) From Remark 2.4, the assumption on the second-order lower semi-differentiability of G at (x, y) in the direction (w, \hat{r}) is superfluous.

(ii) In the proof of the Proposition 2 (see page 248 of [1]), the authors implied that $v - \hat{v} \in D^2\bar{G}(x, \bar{y}, w, \bar{r})(u)$ from the second-order lower semi-differentiability of G . This assertion is right if $\bar{y}_n \in \bar{G}(x_n)$. Since

$$\bar{y}_n = y_n - y_n \in G(x_n) - G(x_n) = \bar{G}(x_n) + G(x_n) - G(x_n)$$

and G is a set-valued map, it is not sure that \bar{y}_n belongs to $\bar{G}(x_n)$. Thus, the conclusion of the Proposition 3.1 may be not true, see Example 2.1 in [11].

Hence, the Proposition 3.1 should be presented as follows.

Proposition 3.2. *Let $\bar{G}, G : X \rightarrow 2^Y$, $(x, \bar{y}) \in gr(\bar{G}), (x, y) \in gr(G)$ and $(w, \bar{r}, \hat{r}) \in X \times Y \times Y$. Suppose that G is second-order semi-differentiable at x relative to y in the direction (w, \hat{r}) . Then, for all $u \in X$,*

$$D^2\bar{G}(x, \bar{y}, w, \bar{r})(u) + D^2G(x, y, w, \hat{r})(u) \subseteq D^2G(x, y, w, r)(u), \tag{2}$$

where $G := \bar{G} + G$, $y := \bar{y} + y$ and $r := \bar{r} + \hat{r}$.

When G is a single-valued map, (2) become an equality. If, additionally, \bar{G} is second-order proto-differentiable at x relative to \bar{y} in the direction (w, \bar{r}) , then G is second-order proto-differentiable at x relative to y in the direction (w, r) .

Proof It is similar to the proof of Proposition 2 in [1].

As an application of Proposition 3.1 to sensitivity analysis of generalized perturbation maps, Theorem 1 in [1] should be stated by

Theorem 3.1. *Let X, Y, Z be normed spaces, $G : X \times Z \rightarrow 2^Y$ be defined by $G(x, z) := \{y \in D \mid z \in F(x, y) + K(y)\}$, where D is a closed convex subset in Y , $F : X \times Y \rightarrow 2^Z$,*

$K : D \rightarrow 2^Z, (y, \bar{z}) \in gr(K), (x, y), \bar{z} - z \in gr(F)$ and $z - \bar{z}$ in the direction $(w, r, q - \bar{q})$. Then, for all $u \in X$ $(w, r, q - \bar{q}) \in X \times (D - y) \times Z$. Suppose that F is second-order semi-differentiable at (x, y) relative to

$$\left\{ v \in T^2(D, y, r) \mid p \in D^2F(x, y, z - \bar{z}, w, r, q - \bar{q})(u, v) + D^2K(y, \bar{z}, r, \bar{q})(v) \right\} \subseteq D^2G(x, z, y, w, q, r)(u, p). \tag{3}$$

When F is a single-valued map, (3) is an equality. If, additionally, K is second-order proto-differentiable at y relative to \bar{z} in the direction (r, \bar{q}) , then G is second-order proto-differentiable at (x, z) relative to y in the direction (w, q, r) .

Proof. The reader is referred to Theorem 1 in [1].

A natural question arises: which conditions ensure that (3.1) becomes an equality when all maps are set-valued? To get the answer, we recall a concept of the TP-derivative (see [12]) of a set-valued map $F : X \rightarrow 2^Y$ at $(x, y) \in gr(F)$ as follows

$$D_{TP}F(x, y)(u) := \left\{ v \in Y \mid \exists t_n > 0, \exists (u_n, v_n) \rightarrow (u, v) : x + t_n u_n \rightarrow x, y + t_n v_n \in F(x + t_n u_n) \right\}.$$

The following concept is necessary for our next result.

Definition 3.1. Let $F : X \rightarrow 2^Y, (x, y) \in gr(F)$ and $(w, r) \in X \times Y$. The asymptotic second-order TP-derivative of F at (x, y) in the direction (w, r) is a set-valued map $D_{TP}^2F(x, y, w, r) : X \rightarrow 2^Y$ defined by

$$D_{TP}^2F(x, y, w, r)(u) := \left\{ v \in Y \mid \begin{array}{l} \exists t_n, h_n > 0, \exists (u_n, v_n) \rightarrow (u, v) : x + t_n w + \frac{t_n h_n}{2} u_n \rightarrow x, \\ y + t_n r + \frac{t_n h_n}{2} v_n \in F\left(x + t_n w + \frac{t_n h_n}{2} u_n\right) \end{array} \right\}.$$

It is obvious to see that $D_{TP}^2F(x, y, 0, 0)(u) = D_{TP}F(x, y)(u)$ for all $u \in X$. By virtue of the asymptotic second-order TP-derivative, we obtain the converse inclusion of (3.1) for set-valued maps as follows.

Proposition 3.3. Let $\bar{G}, G : X \rightarrow 2^Y, (x, \bar{y}) \in gr(\bar{G}), (x, y) \in gr(G)$ and $(w, \bar{r}, \hat{r}) \in X \times Y \times Y$. Suppose that Y is finite dimensional and

$$D_{TP}^2\bar{G}(x, \bar{y}, w, \bar{r})(0) \cap \left(-D_{TP}^2G(x, y, w, \hat{r})(0) \right) = \{0\}. \tag{4}$$

Then, for all $u \in X$,

$$D^2G(x, y, w, r)(u) \subseteq D^2\bar{G}(x, \bar{y}, w, \bar{r})(u) + D^2G(x, y, w, \hat{r})(u),$$

where $G := \bar{G} + G, y := \bar{y} + y$ and $r := \bar{r} + \hat{r}$.

Proof. Let $v \in D^2G(x, y, w, r)(u)$, then there exist $t_n \rightarrow 0^+, (u_n, v_n) \rightarrow (u, v)$ such that

$$y + t_n r + \frac{t_n^2}{2} v_n \in G\left(x + t_n w + \frac{t_n^2}{2} u_n\right) = \bar{G}\left(x + t_n w + \frac{t_n^2}{2} u_n\right) + G\left(x + t_n w + \frac{t_n^2}{2} u_n\right).$$

Thus, there are $\bar{y}_n \in \bar{G}\left(x+t_n w+\frac{t_n^2}{2} u_n\right)$ and $y_n \in G\left(x+t_n w+\frac{t_n^2}{2} u_n\right)$ such that

$$v_n = \bar{v}_n + v_n, \tag{5}$$

where $\bar{v}_n := \frac{\bar{y}_n - \bar{y} - t_n \bar{r}}{(1/2)t_n^2}, v_n := \frac{y_n - y - t_n \hat{r}}{(1/2)t_n^2}$. We now prove that the sequence v_n has a convergent subsequence.

Suppose to the contrary, i.e., $\|v_n\| \rightarrow +\infty$. By setting $z_n := v_n / \|v_n\|$, then z_n (taking a subsequence if necessary) has a limit point \hat{z} with $\|\hat{z}\| = 1$. Moreover,

$$y+t_n \hat{r}+\frac{t_n\left(t_n\|v_n\|\right)}{2} z_n=y_n \in G\left(x+t_n w+\frac{t_n\left(t_n\|v_n\|\right)}{2}\left(\frac{u_n}{\|v_n\|}\right)\right).$$

Let $h_n := t_n \|v_n\|$, we get $\hat{z} \in D_{rp}^2 G(x, y, w, \hat{r})(0)$. It follows from (5) that

$$\frac{v_n}{\|v_n\|}=\frac{\bar{v}_n}{\|v_n\|}+\frac{v_n}{\|v_n\|},$$

which implies $\bar{v}_n / \|v_n\| \rightarrow -\hat{z}$. On the other hand,

$$\bar{y}+t_n \bar{r}+\frac{t_n\left(t_n\|v_n\|\right)}{2}\left(\frac{\bar{v}_n}{\|v_n\|}\right)=\bar{y}_n \in \bar{G}\left(x+t_n w+\frac{t_n\left(t_n\|v_n\|\right)}{2}\left(\frac{u_n}{\|v_n\|}\right)\right).$$

Thus, $-\hat{z} \in D_{rp}^2 \bar{G}(x, \bar{y}, w, \bar{r})(0)$, which contradicts (4).

Without loss of generality, we assume that v_n converges to \hat{v} , then $\hat{v} \in D^2 G(x, y, w, \hat{r})(u)$. From (5),

one gets $\bar{v}_n = v_n - v_n \rightarrow v - \hat{v}$. Hence $v - \hat{v} \in D^2 \bar{G}(x, \bar{y}, w, \bar{r})(u)$, i.e.,

$$v \in D^2 \bar{G}(x, \bar{y}, w, \bar{r})(u)+D^2 G(x, y, w, \hat{r})(u).$$

Note that (3.3) is only a sufficient condition (not necessary condition) for the converse inclusion of (2), see Example 2 in [1].

It follows from the Proposition 3.3 that the Proposition 3.2 and the Theorem 3.1 can be modified for set-valued maps as follows.

Proposition 3.4. *Let Y be finite dimensional, $\bar{G}, G: X \rightarrow 2^Y, (x, \bar{y}) \in \text{gr}(\bar{G}), (x, y) \in \text{gr}(G)$, and $(w, \bar{r}, \hat{r}) \in X \times Y \times Y$. Suppose that G is second-order semi-differentiable at x relative to y in the direction (w, \hat{r}) and (4) holds. Then, (2) becomes an equality. If, additionally, \bar{G} is second-order proto-differentiable at x relative to \bar{y} in the direction (w, \bar{r}) , then G is second-order proto-differentiable at x relative to y in the direction (w, r) .*

Theorem 3.2. *Let Y be finite dimensional, $G: X \times Z \rightarrow 2^Y$ be defined by $G(x, z) := \{y \in D \mid z \in F(x, y) + K(y)\}$, where D is a closed convex subset in $Y, F: X \times Y \rightarrow 2^Z$,*

$K : D \rightarrow 2^Z, (y, \bar{z}) \in gr(K), ((x, y), \bar{z} - z) \in gr(F)$
 and $(w, r, q - \bar{q}) \in X \times (D - y) \times Z$. Suppose that F is
 second-order semi-differentiable at (x, y) relative to
 $z - \bar{z}$ in the direction $(w, r, q - \bar{q})$ and
 $D_{TP}^2 F(x, y, z - \bar{z}, w, r, q - \bar{q})(0, 0) \cap (-D_{TP}^2 K(y, \bar{z}, r, \bar{q})(0)) = \{\emptyset\}$

Then, (3) is an equality. If, additionally, K is second-
 order proto-differentiable at y relative to \bar{z} in the
 direction (r, \bar{q}) , then G is second-order proto-
 differentiable at (x, z) relative to y in the direction
 (w, q, r) .

By taking $(w, r) = (0, 0)$, the Proposition 3.4 and
 the Theorem 3.2 were reduced to Proposition 2.1 and
 the Theorem 3.1 in [11], respectively.

In [1], the second semi-differentiability is
 employed to get (3.1). Although it is a quite strict
 condition, this concept (or relative versions) is used in

$$\bar{y} + t_n \bar{r} + \frac{t_n^2}{2} \bar{v}_n \in \bar{G} \left(x + t_n w + \frac{t_n^2}{2} u_n \right).$$

Since G is second-order proto-differentiable at x relative to y in the direction (w, \hat{r}) , with t_n above, there are
 $(u_n, v_n) \rightarrow (u, \hat{v})$ such that

$$y + t_n \hat{r} + \frac{t_n^2}{2} v_n \in G \left(x + t_n w + \frac{t_n^2}{2} u_n \right).$$

According to the metric subregularity of g , there exist $\mu, \lambda > 0$ such that for every $(u_1, v_1, u_2, v_2) \in$
 $(B_{X \times Y}((x, \bar{y}), \lambda) \times B_{X \times Y}((x, y), \lambda)) \cap (gr(\bar{G}) \times gr(G))$,

$$d \left((u_1, v_1, u_2, v_2), g^{-1}(0_X) \cap (gr(\bar{G}) \times gr(G)) \right) \leq \mu d(0_X, g(u_1, v_1, u_2, v_2)). \tag{6}$$

For n large enough, we have

$$\left(x + t_n w + \frac{t_n^2}{2} u_n, y + t_n \bar{r} + \frac{t_n^2}{2} \bar{v}_n, x + t_n w + \frac{t_n^2}{2} u_n, y + t_n \hat{r} + \frac{t_n^2}{2} v_n \right) \in B_{X \times Y}((x, \bar{y}), \lambda) \times B_{X \times Y}((x, y), \lambda).$$

Thus, it follows from (3.5) that there exists $(\bar{x}_n, \bar{y}_n, x_n, y_n) \in gr(\bar{G}) \times gr(G)$ with $\bar{x}_n = x_n$ for all n such that

recent papers for calculus of several kinds of
 generalized derivatives, see [13-16]. In this paper, we
 propose another hypothesis to obtain (2) without the
 semi-differentiability as follows.

Proposition 3.5. Let $\bar{G}, G : X \rightarrow 2^Y$,
 and

$(w, \bar{r}, \hat{r}) \in X \times Y \times Y$. Suppose that G is second-order
 proto-differentiable at x relative to y in the direction
 (w, \hat{r}) and the map $g : (X \times Y)^2 \rightarrow X$ defined by
 $g(\alpha, \beta, \gamma, \delta) := \alpha - \gamma$ is metric subregular at
 $(x, \bar{y}, x, y, 0_X)$ with respect to $gr(\bar{G}) \times gr(G)$. Then,
 (3.1) holds.

Proof. Let $\bar{v} \in D^2 \bar{G}(x, \bar{y}, w, \bar{r})(u)$ and
 $\hat{v} \in D^2 G(x, y, w, \hat{r})(u)$, then there exist
 $t_n \rightarrow 0^+, (u_n, v_n) \rightarrow (u, \bar{v})$ such that

$$\left\| \left(x, \bar{y}, x, y \right) + t_n \left(w, \bar{r}, w, \hat{r} \right) + \frac{t_n^2}{2} \left(\bar{u}_n, \bar{v}_n, u_n, v_n \right) - \left(\bar{x}_n, \bar{y}_n, x_n, y_n \right) \right\| \leq \mu \frac{t_n^2}{2} \left\| \bar{u}_n - u_n \right\|,$$

which implies

$$\begin{aligned} & \left\| \left(\bar{y} + t_n \bar{r} + \frac{t_n^2}{2} \bar{v}_n - y_n \right) + \left(y + t_n \hat{r} + \frac{t_n^2}{2} v_n - y_n \right) \right\| \\ & \leq \left\| \bar{y} + t_n \bar{r} + \frac{t_n^2}{2} \bar{v}_n - y_n \right\| + \left\| y + t_n \hat{r} + \frac{t_n^2}{2} v_n - y_n \right\| \\ & \leq 2 \left\| \left(x, \bar{y}, x, y \right) + t_n \left(w, \bar{r}, w, \hat{r} \right) + \frac{t_n^2}{2} \left(\bar{u}_n, \bar{v}_n, u_n, v_n \right) - \left(\bar{x}_n, \bar{y}_n, x_n, y_n \right) \right\| \\ & \leq \mu t_n^2 \left\| \bar{u}_n - u_n \right\| \end{aligned}$$

and

$$\left\| x + t_n w + \frac{t_n^2}{2} \bar{u}_n - x_n \right\| \leq \left\| \left(x, \bar{y}, x, y \right) + t_n \left(w, \bar{r}, w, \hat{r} \right) + \frac{t_n^2}{2} \left(\bar{u}_n, \bar{v}_n, u_n, v_n \right) - \left(\bar{x}_n, \bar{y}_n, x_n, y_n \right) \right\| \leq \mu \frac{t_n^2}{2} \left\| \bar{u}_n - u_n \right\|.$$

Thus,

$$\left\| \frac{\left(\bar{y}_n + y_n \right) - \left(\bar{y} + y \right) - t_n \left(\bar{r} + \hat{r} \right)}{\left(1/2 \right) t_n^2} - \left(\bar{v}_n + v_n \right) \right\| \leq 2\mu \left\| \bar{u}_n - u_n \right\| \tag{7}$$

and

$$\left\| \frac{\bar{x}_n - x - t_n w}{\left(1/2 \right) t_n^2} - \bar{u}_n \right\| \leq \mu \left\| \bar{u}_n - u_n \right\|. \tag{8}$$

Setting

$$v_n := \frac{\left(\bar{y}_n + y_n \right) - \left(\bar{y} + y \right) - t_n \left(\bar{r} + \hat{r} \right)}{\left(1/2 \right) t_n^2}, u_n := \frac{\bar{x}_n - x - t_n w}{\left(1/2 \right) t_n^2},$$

then $v_n \rightarrow \bar{v} + \hat{v}, u_n \rightarrow u$ (let $n \rightarrow +\infty$ in (7) and (8) and

$$y + t_n r + \frac{t_n^2}{2} v_n = \bar{y}_n + y_n \in \bar{G}(\bar{x}_n) + G(x_n) = G\left(x + t_n w + \frac{t_n^2}{2} \bar{u}_n\right),$$

where $y := \bar{y} + y$ and $r := \bar{r} + \hat{r}$. Hence, $\bar{v} + \hat{v} \in D^2G(x, y, w, r)(u)$.

Taking $(w, \bar{r}, \hat{r}) = (0, 0, 0)$, the Proposition 3.5 reduces to the following result.

Corollary 3.1. Let $\bar{G}, G: X \rightarrow 2^Y$, $(x, \bar{y}) \in gr(\bar{G}), (x, y) \in gr(G)$. Suppose that G is proto-differentiable at x relative to y and the map $g: (X \times Y)^2 \rightarrow X$ defined as in Proposition 3.5 is metric subregular at $(x, \bar{y}, x, y, 0_x)$ with respect to $gr(\bar{G}) \times gr(G)$. Then,

$$D\bar{G}(x, \bar{y})(u) + DG(x, y)(u) \subseteq D(\bar{G} + G)(x, \bar{y} + y)(u).$$

Corollary 3.1 can be used to replace the semi-differentiability of G in Proposition 2.1 in [11].

Our condition on the metric subregularity in the Proposition 3.5 is very different from the semi-differentiability condition. However, the following example shows a case where the Corollary 3.1 works, while the Proposition 3.2 does not.

Example 3.1. Let $\bar{G}, G: \mathbb{R} \rightarrow 2^{\mathbb{R}}$ be defined by

$$\bar{G}(x) = G(x) := \begin{cases} \emptyset, & x \in \left\{ \frac{1}{n^2} \mid n \in \mathbb{N} \right\}, \\ \{x\}, & \text{otherwise.} \end{cases}$$

Then, we have

$$(\bar{G} + G)(x) := \begin{cases} \emptyset, & x \in \left\{ \frac{1}{n^2} \mid n \in \mathbb{N} \right\}, \\ \{2x\}, & \text{otherwise.} \end{cases}$$

By calculating, one gets

$$DG(0, 0)(u) = D^b G(0, 0)(u) = \{u\}, D_i G(0, 0)(0) = \emptyset,$$

which implies that G is proto-differentiable at 0 relative to 0, but it is not semi-differentiable at 0 relative to 0. Thus, Proposition 3.2 cannot be employed in this example. However, the metric subregularity of Corollary 3.1 is fulfilled.

Indeed, let $\lambda > 0$, we need to show that there exists $\mu > 0$ such that for all $(\bar{u}, \bar{v}', u, \hat{v}') \in (B_{\mathbb{R}^2}((0, 0), \lambda) \times B_{\mathbb{R}^2}((0, 0), \lambda)) \cap (gr(\bar{G}) \times gr(G))$,

$$d\left((\bar{u}, \bar{v}', u, \hat{v}'), g^{-1}(0) \cap (gr(\bar{G}) \times gr(G))\right) \leq \mu d\left(0, g(\bar{u}, \bar{v}', u, \hat{v}')\right).$$

Since $(\bar{u}, \bar{v}', u, \hat{v}') \in gr(\bar{G}) \times gr(G)$, we get $\bar{u} = \bar{v}'$ and $u = \hat{v}'$. Thus, it is enough to find $\mu > 0$ such that

$$\inf_{x \in \mathbb{R}, (\bar{y}, y) \in (\bar{G} \times G)(x)} \left\{ |\bar{u} - x| + |\bar{v}' - \bar{y}| + |u - x| + |\hat{v}' - y| \right\} \leq \mu |\bar{u} - u|. \tag{9}$$

On the other hand, we have

$$\inf_{x \in \mathbb{R}, (\bar{y}, y) \in (\bar{G} \times G)(x)} \left\{ |\bar{u} - x| + |\bar{v}' - \bar{y}| + |u - x| + |\hat{v}' - y| \right\} = 2 \inf_{x \in \mathbb{R}} \left\{ |\bar{u} - x| + |u - x| \right\}.$$

Taking $x = \frac{\bar{u} + u}{2}$, then we get

$$2 \inf_{x \in \mathbb{R}} \left\{ \left| \bar{u} - x \right| + \left| u - x \right| \right\} \leq 2 \left(\left| \bar{u} - \frac{\bar{u} + u}{2} \right| + \left| u - \frac{\bar{u} + u}{2} \right| \right) = 2 \left| \bar{u} - u \right|.$$

Thus, (9) is true for every $\mu \geq 2$. Hence, by Corollary 3.1, we get

$$D\bar{G}(0,0)(u) + DG(0,0)(u) \subseteq D(\bar{G} + G)(0,0)(u).$$

CONCLUSION

In the paper, we propose remarks for some results in [1]. Then, by virtue of the proto-differentiability, a weaker hypothesis than the semi-differentiability introduced in [1], we obtain a new result on second-

order sensitivity analysis of generalized perturbation maps.

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Đạo hàm contingent cấp hai của ánh xạ nhiều suy rộng

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TÓM TẮT

Trong bài báo này, chúng tôi đưa ra một số nhận xét về các kết quả của [1]. Sau đó, chúng tôi phát triển các kết quả chính liên quan đến phép tổng của đạo hàm contingent cấp hai cho ánh xạ đa trị và các áp dụng

trong phân tích độ nhạy của ánh xạ nhiều suy rộng. Kết quả đạt được là mới và cải thiện các kết quả của [1]. Một số ví dụ minh họa cũng được đưa ra.

Từ khoá: Tiền khả vi cấp hai, nửa khả vi cấp hai, chính quy dưới metric, ánh xạ nhiều suy rộng, ánh xạ đa trị

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