General solutions of the theme “Light propagation in optical uniaxial crystals”

- **Truong Quang Nghia**
- **Nguyen Tu Ngoc Huong**

University of Science, Vietnam National University-Ho Chi Minh City

(Received on Deceember 13th 2016, accepted on July 26th 2017)

ABSTRACT

In this article, we introduce a new approach to receive general solutions which describe all of the properties of the light propagating across optical uniaxial crystals. In our approach we do not use the conception of refractive index ellipsoid as being done in references. The solutions are given in analytical expressions so we can handly calculate or writing a small program to compute these expressions.

Keywords: *extra-ordinary ray, light polarization, light velocity, Maxwell’s equations, optical uniaxial crystals, ordinary ray, refractive index, tensor*

INTRODUCTION

The problem of lights propagation in optical uniaxial crystals, i.e. crystals of trigonal, tetragonal and hexagonal systems, was solved by the application of Maxwell’s equations. Solving the Maxwell’s equations for a plane wave light propagating in transparent non-magnetic crystals, one can derive two refractive indices of the two propagating modes of light [2, 3]:

$$n_{o,e}^{-2} = \frac{1}{2}\left[\left(\eta_{11}+\eta_{22}\right) \pm \sqrt{\left(\eta_{11}-\eta_{22}\right)^2 + 4\eta_{12}^2}\right] \qquad (1)$$

In (1), η_{ij} (i, j = 1, 2) are the components of the dielectric impermeability tensor of crystal. In expression (1), the light direction is taken in parallel to axis OX_3 of an arbitrary coordinate axes OX_i (i = 1, 2, 3).

Unfortunately, in the reality it is difficult to use the general expression (1) to receive two refractive indices, because in references the components of tensor [η_{ij}] are often given in crystal coordinate axes OX_i^* (i = 1, 2, 3) where the number of independent components of this tensor is minimum, i.e. η_{11}^* and η_{33}^* (for optical uniaxial crystals).

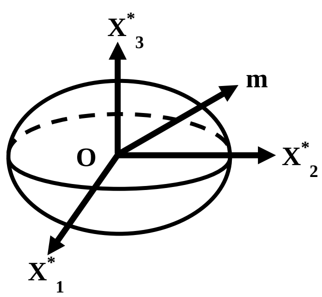


Fig 1. The crystal coordinate axes OX_i^* (i = 1, 2, 3) and the light direction m

On the other side, when the light direction varies, the components η_{11}, η_{22} and η_{12} in (1) also vary in according to the light direction. Therefore in references, in order to eliminate this difficulty, one can only solve this problem in crystal coordinate axes OX_i^* with the help of the conception of refractive index ellipsoid, but this approach can only be applied in some limited cases when light propagating in some special symmetric directions of the crystal. The refractive index ellipsoid of optical uniaxial crystals is an ellipsoid of revolution. It has an important property: the central section perpendicular to the light direction $\boldsymbol{m} = \left(m_1\ ,\ m_2\ ,m_3\right)$ is an ellipse and the refractive indices of the two waves are given by the lengths of the semi–axes of this ellipse and the directions of these semi–axes give the directions of oscillations of the eigen vectors $\boldsymbol{D}^{*(o)}$ and $\boldsymbol{D}^{*(e)}$ for each of the two modes of light.

By this approach, it is difficult to solve the problem when light propagating in an arbitrary m direction. In order to eliminate this difficulty, in this article we introduce a new approach using the general solution (1). Here, the important query is the calculation of the components η_{11}, η_{22} and η_{12} via the components η_{11}^* and η_{33}^* given in crystal coordinate axes OX_i^*. In order to do that we have to find the transformation cosinus matrix $\left(\alpha_i^k\right)$ (i, k = 1, 2, 3) of the transformation of axes from OX_i^* to OX_i. Having found $\left(\alpha_i^k\right)$ we apply the transformation rule of the components of a second rank tensor $[\eta_{ij}^*]$ to derive the corresponding components η_{ij} in an arbitrary coordinate axes OX_i. Replacing observed values of η_{11}, η_{22} and η_{12} into general expression (1) we can solve the given problem.

THEORETICAL CALCULATIONS

The transformation cosinus matrix $\left(\alpha_i^k\right)$

In an arbitrary coordinate axes OX_i (i = 1, 2, 3) we choose the OX_3 axis which is parallel to the light direction m, which has three components (cosinus) in crystal coordinate axes: m_1 ; m_2 ; m_3. Thus, the unit vector $\boldsymbol{u}^{(3)}$ along axis equals to m

$$\boldsymbol{u}^{(3)} = \boldsymbol{m} = \left(m_1\ ,\ m_2\ ,\ m_3\right) \qquad (2)$$

Denoting g and h two vectors (not unit vectors) prolonging axes OX_1 , OX_2 respectively.

We can write:

$$\boldsymbol{g} = \left(1\ ,\ 0\ ,\ 0\right) + \mu \boldsymbol{u}^{(3)}$$

Where (1 , 0 , 0) is the unit vector along OX_1^* and μ is a coefficient derived from the orthogonal condition $\boldsymbol{g}.\boldsymbol{u}^{(3)} = 0$.

Applying this orthogonal condition $\boldsymbol{g}.\boldsymbol{u}^{(3)} = \left[\left(1\ ,\ 0\ ,\ 0\right) + \mu \boldsymbol{u}^{(3)}\right].\boldsymbol{u}^{(3)} = 0$

We find $\mu = -m_1$.

Thus,

$$\boldsymbol{g} = \left(1\ ,\ 0\ ,\ 0\right) - m_1 \boldsymbol{u}^{(3)} = \left[\left(1 - m_1^2\right)\ ,\ -m_1 m_2\ ,\ -m_1 m_3\right] \qquad (3)$$

are the components of g in axes OX_1^*, OX_2^* and OX_3^* respectively.

The vector h along the axis OX_2 can be written in the form:

$$\boldsymbol{h} = \left(0\ ;\ 1\ ;\ 0\right) + \mu_1 \boldsymbol{g} + \mu_2 \boldsymbol{u}^{(3)}$$

Where (0 , 1 , 0) is an unit vector along OX_2^*, μ_1 and μ_2 are the coefficients derived from the orthogonal conditions $\boldsymbol{h}.\boldsymbol{g} = \boldsymbol{h}.\boldsymbol{u}^{(3)} = 0$.

From these orthogonal conditions we find: $\mu_2 = -m_2$ **và** $\mu_1 = \dfrac{m_1 m_2}{1 - m_1^2}$

Thus, $$\boldsymbol{h} = \left(0\ ;\ 1\ ;\ 0\right) + \frac{m_1 m_2}{1 - m_1^2} \boldsymbol{g} - m_2 \boldsymbol{u}^{(3)} \qquad (4)$$

From expressions (2), (3), (4) we can derive the components of h along the axes OX_i^* (i = 1, 2, 3):

$$\boldsymbol{h} = \left[\frac{m_1 m_2}{1 - m_1^2}\left(1 - m_1^2\right) - m_2 m_1\ ,\ 1 - \frac{m_1^2 m_2}{1 - m_1^2} - m_2^2\ ,\ \frac{-m_1^2 m_2 m_3}{1 - m_1^2} - m_2 m_3\right] = \left[0\ ,\ \frac{m_3^2}{1 - m_1^2}\ ,\ \frac{m_2 m_3}{1 - m_1^2}\right]$$

The components of g and h along the OX_i^* are not the direction cosinus of axes OX_1, OX_2 versus OX_1^*, OX_2^*, OX_3^*, but these direction cosinus can be derived by dividing these components by their vector length, i.e. $|\boldsymbol{g}|$ and $|\boldsymbol{h}|$.

Finally, we obtain the direction cosinus matrix $\left(\alpha_i^k\right)$ of the transformation of axes from OX_i^* to OX_i :

$$\left(\alpha_i^k\right)=\begin{pmatrix} \sqrt{1-m_1^2} & \dfrac{-m_1m_2}{\sqrt{1-m_1^2}} & \dfrac{-m_1m_3}{\sqrt{1-m_1^2}} \\ 0 & \dfrac{m_3}{\sqrt{1-m_1^2}} & \dfrac{-m_2}{\sqrt{1-m_1^2}} \\ m_1 & m_2 & m_3 \end{pmatrix} \quad (5)$$

We can verify the truth of this matrix by these tests:

* $\left(\alpha_i^1\right)^2+\left(\alpha_i^2\right)^2+\left(\alpha_i^3\right)^2=1$ (i =1, 2, 3)

* The orthogonal conditions of axes OX_i (i =1, 2, 3) via their scalar multiplications.

* The determinant of matrix $\left(\alpha_i^k\right)$ must be equal to 1 if the observed coordinate axes OX_i form a right – handed system.

The components of dielectric impermeability tensors [η_{ij}] in coordinate axes OX_i

Applying the transformation rule of the components of a second rank tensor when the coordinate axes varies from OX_i^* to OX_i : [1]

$$\eta_{ij}=\alpha_i^k\alpha_j^\ell\eta_{k\ell}^* \quad \text{(i, j, k, ℓ = 1, 2, 3)} \qquad (6)$$

In expression (6) we used Einstein notation, i.e. to take the summation of the repeated indices by running this index from 1 to 3.

For example:

$$\eta_{11}=\alpha_1^k\alpha_1^\ell\eta_{k\ell}^*=\alpha_1^k\left(\alpha_1^1\eta_{k1}^*+\alpha_1^2\eta_{k2}^*+\alpha_1^3\eta_{k3}^*\right)$$

$$=\alpha_1^1\left(\alpha_1^1\eta_{11}^*+\alpha_1^2\eta_{12}^*+\alpha_1^3\eta_{13}^*\right)+\alpha_1^2\left(\alpha_1^1\eta_{21}^*+\alpha_1^2\eta_{22}^*+\alpha_1^3\eta_{23}^*\right)+\alpha_1^3\left(\alpha_1^1\eta_{31}^*+\alpha_1^2\eta_{32}^*+\alpha_1^3\eta_{33}^*\right)$$

$$=\left(\alpha_1^1\right)^2\eta_{11}^*+\left(\alpha_1^2\right)^2\eta_{11}^*+\left(\alpha_1^3\right)^2\eta_{33}^*=\left[\left(\alpha_1^1\right)^2+\left(\alpha_1^2\right)^2\right]\eta_{11}^*+\left(\alpha_1^3\right)^2\eta_{33}^*$$

$$=\eta_{11}^*+\frac{m_1^2m_3^2}{1-m_1^2}\left(\eta_{33}^*-\eta_{11}^*\right)$$

In this example we have taken into account tensor [η_{ij}^*] is diagonal for optical uniaxial crystals and $\eta_{11}^*=\eta_{22}^*$ and $\left(m_1^2+m_2^2+m_3^2\right)=1$.

Analogously, we can derive all the components of tensor [η_{ij}] in the coordinate axes OX_i as follows:

$$\left[\eta_{ij}\right]=\begin{bmatrix} \eta_{11}^*+\dfrac{m_1^2m_3^2}{1-m_1^2}\left(\eta_{33}^*-\eta_{11}^*\right) & \dfrac{m_1m_2m_3}{1-m_1^2}\left(\eta_{33}^*-\eta_{11}^*\right) & \dfrac{-m_1m_3^2}{\sqrt{1-m_1^2}}\left(\eta_{33}^*-\eta_{11}^*\right) \\ \eta_{12} & \dfrac{m_3^2}{1-m_1^2}\eta_{11}^*+\dfrac{m_2^2}{1-m_1^2}\eta_{33}^* & \dfrac{-m_2m_3}{\sqrt{1-m_1^2}}\left(\eta_{33}^*-\eta_{11}^*\right) \\ \eta_{13} & \eta_{23} & \eta_{11}^*+m_3^2\left(\eta_{33}^*-\eta_{11}^*\right) \end{bmatrix} \quad (7)$$

Now, replacing η_{11}, η_{12} and η_{22} from (7) into the general expression (1) we can solve the proposed problem.

After a long way of calculations we derive the refractive indices for the two propagating modes of light:

$$n_{o,e}^{-2}=\frac{1}{2}\left[\left(1+m_3^2\right)\eta_{11}^*+\left(1-m_3^2\right)\eta_{33}^*\pm\left(1-m_3^2\right)\left(\eta_{33}^*-\eta_{11}^*\right)\right] \tag{8}$$

The corresponding refractive index of ordinary and extra-ordinary rays

Now here, we discuss what of the sign (+ or –) in (8) of which the refractive index of ordinary ray will be taken. For the convenience of discussion we rewrite expression (8) in the form:

$$n_{o,e}^{-2}=\frac{1}{n_{o,e}^2}=\frac{1}{2}\left[\left(1+m_3^2\right)\eta_{11}^*+\left(1-m_3^2\right)\eta_{33}^*\pm B\right]=A_{o,e}\,;\; n_{o,e}^2=\frac{1}{A_{o,e}} \text{ and } B=\left(1-m_3^2\right)\left(\eta_{33}^*-\eta_{11}^*\right)$$

There are two cases for discussion:

* Positive optical crystals $(n_e > n_o)$ or $\left(\eta_{33}^*-\eta_{11}^*\right)<0$

In this case, because of the refractive index of ordinary ray $n_o < n_e$, the quantity A_o must be greater. On the other side, in this case $\left(\eta_{33}^*-\eta_{11}^*\right)<0$ so that $-B>0$. Thus n_o^{-2} takes the sign (–) and therefore n_e^{-2} takes the sign (+) in expression (8).

* Negative optical crystals $(n_e < n_o)$ or $\left(\eta_{33}^*-\eta_{11}^*\right)>0$

In this case, $B>0$ and A must be smaller so n_o^{-2} also takes the sign (–) and n_e^{-2} takes the sign (+).

Finally, regardless of positive or negative optical crystals, the refractive index of ordinary and extra-ordinary rays have the expressions:

$$\left.\begin{aligned} n_o^{-2}&=\frac{1}{2}\left[\left(1+m_3^2\right)\eta_{11}^*+\left(1-m_3^2\right)\eta_{33}^*-\left(1-m_3^2\right)\left(\eta_{33}^*-\eta_{11}^*\right)\right]\\ n_e^{-2}&=\frac{1}{2}\left[\left(1+m_3^2\right)\eta_{11}^*+\left(1-m_3^2\right)\eta_{33}^*+\left(1-m_3^2\right)\left(\eta_{33}^*-\eta_{11}^*\right)\right]\end{aligned}\right\} \tag{9}$$

* For ordinary ray:

$$n_o^{-2}=\frac{1}{2}\left[\left(1+m_3^2\right)\eta_{11}^*+\left(1-m_3^2\right)\eta_{33}^*-\left(1-m_3^2\right)\left(\eta_{33}^*-\eta_{11}^*\right)\right]=\frac{1}{2}\left[2\eta_{11}^*\right]=\eta_{11}^*$$

$$n_o=\frac{1}{\sqrt{\eta_{11}^*}} \tag{10}$$

Therefore the velocity of ordinary ray propagating across the crystal:

$$v_o=\frac{c}{n_o}=c\sqrt{\eta_{11}^*} \text{ and is independent of light direction.} \tag{11}$$

* For extra-ordinary ray:

$$n_e^{-2}=\frac{1}{2}\left[\left(1+m_3^2\right)\eta_{11}^*+\left(1-m_3^2\right)\eta_{33}^*+\left(1-m_3^2\right)\left(\eta_{33}^*-\eta_{11}^*\right)\right]=\frac{1}{2}\left[2m_3^2\eta_{11}^*+2\left(1-m_3^2\right)\eta_{33}^*\right]=m_3^2\eta_{11}^*+\left(1-m_3^2\right)\eta_{33}^*$$

$$n_e=\frac{1}{\sqrt{m_3^2\eta_{11}^*+\left(1-m_3^2\right)\eta_{33}^*}} \tag{12}$$

The velocity of extra-ordinary ray:

$$v_e=\frac{c}{n_e}=c\sqrt{m_3^2\eta_{11}^*+\left(1-m_3^2\right)\eta_{33}^*} \tag{13}$$

The polarization of the two rays

Denote $\boldsymbol{D}^{(o)}$ and $\boldsymbol{D}^{(e)}$, the unit vectors of polarization of the two rays in coordinate axes OX_i . Because the light is transversal, so in OX_i :

$$\boldsymbol{D}^{(o)} = \left(D_1^{(o)}\ ,\ D_2^{(o)}\ ,\ 0\right) \text{ and } \boldsymbol{D}^{(e)} = \left(D_1^{(e)}\ ,\ D_2^{(e)}\ ,\ 0\right)$$

In order to derive $\boldsymbol{D}^{(o)}$ and $\boldsymbol{D}^{(e)}$ we have to solve the equations determined the eigen vectors of a two-dimension tensor [η_{ij}] having known eigen values n_o and n_e.

$$\eta_{ij} D_j = n^{-2} D_i = 0 \text{ (i, j = 1, 2)}$$

Using the Kronecker notation δ_{ij} , we can write these above equations in the form:

$$\left(\eta_{ij} - n^{-2}\delta_{ij}\right) D_j = 0 \qquad (14)$$

* For the ordinary ray:

Replacing $n^{-2} = n_o^{-2} = \eta_{11}^*$ from (10) into the equations (14) we have:

$$\begin{cases} \left(\eta_{11} - \eta_{11}^*\right) D_1^{(o)} + \eta_{12} D_2^{(o)} = 0 \\ \eta_{12} D_1^{(o)} + \left(\eta_{22} - \eta_{11}^*\right) D_2^{(o)} = 0 \end{cases}$$

Combining with the normalization of $\boldsymbol{D}^{(o)}$, i.e. $\left|\boldsymbol{D}^{(o)}\right| = 1$ we solve the equations and derive the components of $\boldsymbol{D}^{(o)}$ as follows:

$$\boldsymbol{D}^{(o)} = \left(\frac{m_2}{\sqrt{m_1^2 m_3^2 + m_2^2}}\ ,\ \frac{-m_1 m_3}{\sqrt{m_1^2 m_3^2 + m_2^2}}\ ,\ 0 \right) \qquad (15)$$

* For extra-ordinary ray:

Replacing $n^{-2} = n_e^{-2} = m_3^2 \eta_{11}^* + \left(1 - m_3^2\right)\eta_{33}^*$ from (12) into the equation (14) to determine $\boldsymbol{D}^{(e)}$:

$$\begin{cases} \left[\eta_{11} - \left[m_3^2 \eta_{11}^* + \left(1 - m_3^2\right)\eta_{33}^*\right]\right] D_1^{(e)} + \eta_{12} D_2^{(e)} = 0 \\ \eta_{12} D_1^{(e)} + \left[\eta_{22} - \left[m_3^2 \eta_{11}^* + \left(1 - m_3^2\right)\eta_{33}^*\right]\right] D_2^{(e)} = 0 \end{cases}$$

Combining with the normalized condition of $\boldsymbol{D}^{(e)}$ we derive:

$$\boldsymbol{D}^{(e)} = \left(\frac{m_1 m_3}{\sqrt{m_1^2 m_3^2 + m_2^2}}\ ,\ \frac{m_2}{\sqrt{m_1^2 m_3^2 + m_2^2}}\ ,\ 0 \right) \qquad (16)$$

We can verify the orthogonality of $\boldsymbol{D}^{(o)}$ and $\boldsymbol{D}^{(e)}$ via their scalar product.

The polarization of the rays in crystal coordinate axes OX_i^*

Remember that, the light direction $\boldsymbol{m} = \left(m_1\ ,\ m_2\ ,\ m_3\right)$ was given in crystal coordinate axes OX_i^*, so we have to transform the polarized vectors $\boldsymbol{D}^{(o)}$ and $\boldsymbol{D}^{(e)}$ into their corresponding vectors $\boldsymbol{D}^{*(o)}$ and $\boldsymbol{D}^{*(e)}$ in OX_i^*. The transformation cosinus matrix is now $\left(\beta_i^k\right)$, which is the inverse matrix of $\left(\alpha_i^k\right)$.

Rotating matrix from (5) around its diagonal by an angle π we have:

$$\left(\beta_i^k\right) = \begin{pmatrix} \sqrt{1 - m_1^2} & 0 & m_1 \\ \frac{-m_1 m_2}{\sqrt{1 - m_1^2}} & \frac{m_3}{\sqrt{1 - m_1^2}} & m_2 \\ \frac{-m_1 m_3}{\sqrt{1 - m_1^2}} & \frac{-m_2}{\sqrt{1 - m_1^2}} & m_3 \end{pmatrix} \qquad (17)$$

Applying the transformation rule of the components of a vector:

$$D_i^* = \beta_i^k D_k \quad \text{(i, k = 1, 2, 3)}$$

We derive the vectors $\boldsymbol{D}^{*(o)}$ and $\boldsymbol{D}^{*(e)}$ in the crystal coordinate axes OX_i^*:

$$\boldsymbol{D}^{*(o)} = \left(\frac{m_2}{\sqrt{1-m_3^2}} \ , \ \frac{-m_1}{\sqrt{1-m_3^2}} \ , \ 0 \right) \tag{18}$$

$$\boldsymbol{D}^{*(e)} = \left(\frac{m_1 m_3}{\sqrt{1-m_3^2}} \ , \ \frac{m_2 m_3}{\sqrt{1-m_3^2}} \ , \ -\sqrt{1-m_3^2} \right) \tag{19}$$

From (18) we see that the ordinary ray is always polarized in the plane $\left(OX_1^*,\ OX_2^*\right)$ or the plane perpendicular to OX_3^*, i.e. the optical axis of crystals.

We can verify the truth of (18) and (19) by the following tests:

* The orthogonality of $\boldsymbol{D}^{*(o)}$ and $\boldsymbol{D}^{*(e)}$ via their scalar multiplication.

* The orthogonalities $\boldsymbol{D}^{*(o)}.\boldsymbol{m} = \boldsymbol{D}^{*(e)}.\boldsymbol{m} = 0$

* The normalized conditions of vectors $\boldsymbol{D}^{*(o)}$ and $\boldsymbol{D}^{*(e)}$.

The lack of the coincidence between the light direction m and the direction of light energy transfer P (the Poynting vector)

According to [2], [3] the angle α of the lack of coincidence between the light direction m and the direction of light energy transfer, i.e. Poynting vector P is determined by the following expression:

$cos\alpha = \dfrac{\boldsymbol{E}^*.\boldsymbol{D}^*}{\left|\boldsymbol{E}^*\right|\left|\boldsymbol{D}^*\right|} = \dfrac{\boldsymbol{E}^*.\boldsymbol{D}^*}{\left|\boldsymbol{E}^*\right|}$ if vector $\boldsymbol{D}^*$ is normalized.

Thus, in order to calculate α we have to determine the electric vector $\boldsymbol{E}^*$. Because in crystal coordinate axes OX_i^* the dielectric impermeability tensor [η_{ij}^*] is diagonal, therefore

$$E_i^* = \eta_{ij}^* D_j^* = \eta_{ii}^* D_i^* \quad \text{(i = 1, 2, 3)}$$

For the ordinary ray from (18):

$$\Leftrightarrow \begin{cases} E_1^{*(o)} = \eta_{11}^*.D_1^{*(o)} = \dfrac{m_2}{\sqrt{1-m_3^2}}\eta_{11}^* \\ E_2^{*(o)} = \eta_{22}^*.D_2^{*(o)} = \eta_{22}^*.\dfrac{-m_1}{\sqrt{1-m_3^2}} = \dfrac{-m_1}{\sqrt{1-m_3^2}}\eta_{11}^* \\ E_3^{*(o)} = \eta_{33}^*.D_3^{*(o)} = 0 \end{cases}$$

Therefore : $\left|\boldsymbol{E}^{*(o)}\right| = \eta_{11}^*$

Analogously, for the extra-ordinary ray:

$$\Leftrightarrow \begin{cases} E_1^{*(e)} = \dfrac{m_1 m_3}{\sqrt{1-m_3^2}}\eta_{11}^* \\ E_2^{*(e)} = \dfrac{m_2 m_3}{\sqrt{1-m_3^2}}\eta_{11}^* \\ E_3^{*(e)} = -\sqrt{1-m_3^2}\,\eta_{33}^* \end{cases}$$

Therefore : $\left|\boldsymbol{E}^{*(e)}\right| = \sqrt{m_3^2\left(\eta_{11}^*\right)^2 + \left(1-m_3^2\right)\left(\eta_{33}^*\right)^2}$

* For the ordinary ray:

$$cos\alpha_o = \frac{\boldsymbol{E}^{*(o)}.\boldsymbol{D}^{*(o)}}{\left|\boldsymbol{E}^{*(o)}\right|} = \frac{E_1^{*(o)}.D_1^{*(o)} + E_2^{*(o)}.D_2^{*(o)}}{\eta_{11}^*} = \frac{\eta_{11}^*}{\eta_{11}^*} = 1 \Rightarrow \alpha_o = 0^o \quad (20)$$

Thus, for the ordinary ray, there is no lack of coincidence between m and P.

* For the extra-ordinary ray:

$$cos\alpha_e = \frac{\boldsymbol{E}^{*(e)}.\boldsymbol{D}^{*(e)}}{\left|\boldsymbol{E}^{*(e)}\right|} = \frac{m_3^2\eta_{11}^* + \left(1-m_3^2\right)\eta_{33}^*}{\sqrt{m_3^2\left(\eta_{11}^*\right)^2 + \left(1-m_3^2\right)\left(\eta_{33}^*\right)^2}} \quad (21)$$

Before applying our results to some specific cases, we summarize all the solutions we have derived. In crystal coordinate axes OX_i^*: light direction $\boldsymbol{m} = \left(m_1\ ,\ m_2\ ,\ m_3\right)$

* For the ordinary ray:

+ Refractive index : $n_o = \dfrac{1}{\sqrt{\eta_{11}^*}}$ (22)

+ Light velocity : $v_o = c\sqrt{\eta_{11}^*}$ where c is the light velocity in vacuum. (23)

+ Light polarization : $\boldsymbol{D}^{*(o)} = \left(\dfrac{m_2}{\sqrt{1-m_3^2}}\ ,\ \dfrac{-m_1}{\sqrt{1-m_3^2}}\ ,\ 0\right)$ (24)

+ The ordinary ray is always polarized in the plane perpendicular to the optical axe of crystals.

+ There is a coincidence between m and P.

* For the extra-ordinary ray:

+ Refractive index : $n_e = \dfrac{1}{\sqrt{m_3^2\eta_{11}^* + \left(1-m_3^2\right)\eta_{33}^*}}$ (25)

+ Light velocity : $v_e = c\sqrt{m_3^2\eta_{11}^* + \left(1-m_3^2\right)\eta_{33}^*}$ (26)

+ Light polarization : $\boldsymbol{D}^{*(e)} = \left(\dfrac{m_1 m_3}{\sqrt{1-m_3^2}}\ ,\ \dfrac{m_2 m_3}{\sqrt{1-m_3^2}}\ ,\ -\sqrt{1-m_3^2}\right)$ (27)

+ Angle α_e of lack of coincidence between m and P:

$$cos\alpha_e = \frac{m_3^2\eta_{11}^* + \left(1-m_3^2\right)\eta_{33}^*}{\sqrt{m_3^2\left(\eta_{11}^*\right)^2 + \left(1-m_3^2\right)\left(\eta_{33}^*\right)^2}} \quad (28)$$

APPLICATION

To test the truth of our above results, in the application we use KDP crystal. KDP (Dihydro-Phosphate-Kali: KH_2PO_4) is a crystal of tetragonal system. Its point symmetry group is $\bar{4}2m$. It has an inverse axe $\overline{A_4}$, two axes A_2, which are perpendicular to $\overline{A_4}$, two mirrors M which contain $\overline{A_4}$. The Fig. 2 shows the polar projection and the crystallographic axes of KDP:

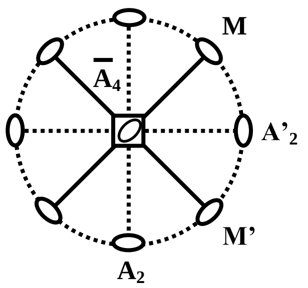
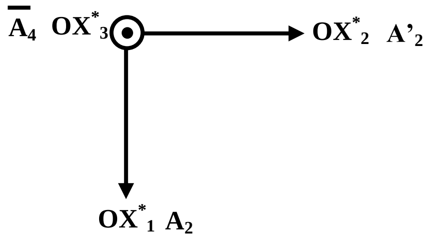


Fig 2. A) Polar projection of point group $\bar{4}2m$ of KDP



B) Crystallographic axes of KDP

$OX_1^* \parallel A_2$; $OX_2^* \parallel A'_2$; $OX_3^* \parallel \overline{A_4}$

In crystallographic coordinate axes OX_i^* the tensor dielectric impermeability of KDP is:

$$\left[\eta_{ij}^*\right] = \begin{bmatrix} 0.43858 & 0 & 0 \\ 0 & 0.43858 & 0 \\ 0 & 0 & 0.46277 \end{bmatrix}$$

Because $\eta_{33}^* = 0.46277 > \eta_{11}^* = 0.43858$, KDP is a negative optical crystal. Its optical axe is the $\overline{A_4}$ axe and in this case is parallel to axis OX_3^*.

We apply our above results in three cases:

The light direction is along the optical axe of KDP

In this case we have $m_1 = m_2 = 0$ and $m_3 = 1$

This is the simplest case of light propagation in optical uniaxial crystals and interestingly to be discussed here. In references, we know that in this case we have only one ray propagating along the optical axe of KDP. This is the ordinary mode. Its polarization can be taken in any direction belonging in the plane perpendicular to the optical axe.

Which, for our results:

* For the ordinary ray:

From (22), (23) we have:

$$n_o = \frac{1}{\sqrt{\eta_{11}^*}} = \frac{1}{\sqrt{0.43858}} = 1.51$$

$$v_o = \frac{c}{n_o} = \frac{300000}{1.51} km/s = 198675.5 \,\text{km/s}$$

From (24) we derive the light polarization:

$$\boldsymbol{D}^{*(o)} = \left(\frac{m_2}{\sqrt{1-m_3^2}} \ , \ \frac{-m_1}{\sqrt{1-m_3^2}} \ , \ 0 \right) = \left(\frac{0}{0} \ , \ \frac{-0}{0} \ , \ 0 \right) \text{th}$$

e polarization of this mode is undetermined. This query will be discussed later.

* For the extra-ordinary ray:

From (25) we have :

$$n_e = \frac{1}{\sqrt{m_3^2 \eta_{11}^* + \left(1-m_3^2\right)\eta_{33}^*}} = \frac{1}{\sqrt{\eta_{11}^*}} = n_o$$

This means that, in this case we have only one mode propagating along optical axe of KDP. It is the ordinary ray.

Light polarization is calculated from (27):

$$\boldsymbol{D}^{*(e)} = \left(\frac{m_1 m_3}{\sqrt{1-m_3^2}} \ , \ \frac{m_2 m_3}{\sqrt{1-m_3^2}} \ , \ -\sqrt{1-m_3^2} \right) = \left(\frac{0}{0} \ , \ \frac{0}{0} \ , \ -0 \right)$$

is also undetermined.

From these above results we see that the polarization of the rays is undetermined but these polarizations are certainly lying in the plane perpendicular to optical axe because $D_3^{*(o)} = D_3^{*(e)} = 0$. The ratio $\left(\frac{0}{0}\right)$ will go to some limitd values, which is not infinity but depends on the light polarization entering the crystal. Imagine a laser beam with any polarization entering along the crystalographic axe of crystal. The polarization of the laser beam can now combine two perpendicular components lying in the plane perpendicular to the optical axe of crystal. Each of the components is the polarization vector for mode n_o or n_e. Although their lengths are not equal to 1, but as shown in [2] the important thing is not the eigen vector but eigen direction as all vectors of arbitrary lengths provided lying along this direction are also eigen vectors of a second rank tensor. Thus, in references we frequently speak about eigen direction instead of the eigen vector. In our case the laser beam will propagate across the crystal with its original polarization. It is the ordinary ray. The laser beam can be polarized in any direction so the plane perpendicular to optical axe of KDP is an eigen plane.

* For the extra-ordinary ray:

$$n_e = \frac{1}{\sqrt{m_3^2\eta_{11}^* + \left(1-m_3^2\right)\eta_{33}^*}} = \frac{1}{\sqrt{\eta_{33}^*}} = \frac{1}{\sqrt{0.46277}} = 1.47$$

$$v_e = \frac{c}{n_e} = \frac{300000}{1.47} km/s = 204081.6\,\text{km/s}$$

Polarization (figure 3) :

$$\boldsymbol{D}^{*(e)} = \left(\frac{m_1 m_3}{\sqrt{1-m_3^2}} , \frac{m_2 m_3}{\sqrt{1-m_3^2}} , -\sqrt{1-m_3^2} \right)$$
$$= (0 , 0 , -1)$$
$$= \left(90^o , 90^o , 180^o\right)$$

Angle of lack of coincidence between m and P:

$$cos\alpha_e = \frac{m_3^2\eta_{11}^* + \left(1-m_3^2\right)\eta_{33}^*}{\sqrt{m_3^2\left(\eta_{11}^*\right)^2 + \left(1-m_3^2\right)\left(\eta_{33}^*\right)^2}}$$
$$= \frac{\eta_{33}^*}{\eta_{33}^*} = 1 \Rightarrow \alpha_e = 0^o$$

There is a coincidence between m and P.

In this case, we can say that we have two ordinary rays propagating with different velocities along an A_2 of KDP.

In conclusion of this discussion, our results are the same already known in the classical approach.

The light direction is along one of the two axes A_2 of KDP (along $\boldsymbol{OX_1^*}$ or $\boldsymbol{OX_2^*}$)

For example, the light direction is $m_1 = 1,\ m_2 = m_3 = 0$

* For the ordinary ray:

$$n_o = \frac{1}{\sqrt{\eta_{11}^*}} = \frac{1}{\sqrt{0.43858}} = 1.51$$

$$v_o = \frac{c}{n_o} = \frac{300000}{1.51} km/s = 198675.5\,\text{km/s}$$

Polarization (Fig 3) :

$$\boldsymbol{D}^{*(o)} = \left(\frac{m_2}{\sqrt{1-m_3^2}} , \frac{-m_1}{\sqrt{1-m_3^2}} , 0 \right)$$
$$= (0 , -1 , 0)$$
$$= \left(90^o , 180^o , 90^o\right)$$

There is a coincidence between m and P : $\alpha_o = 0^o$

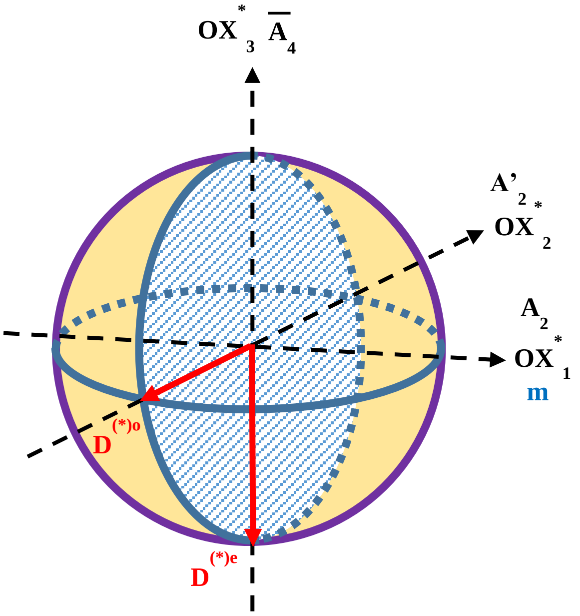


Fig 3. The polarization of the ordinary ray and extra-ordinary ray.

The light direction $\boldsymbol{m} = \left(\frac{1}{\sqrt{6}}\ ,\ \frac{2}{\sqrt{6}}\ ,\ \frac{1}{\sqrt{6}}\right) = \left(65.91^o\ ,\ 35.26^o\ ,\ 65.91^o\right)$

In this case it is difficult to use the refractive index ellipsoid approach to solve the problem.
Our results:

* For the ordinary ray:

$$n_o = \frac{1}{\sqrt{\eta_{11}^*}} = \frac{1}{\sqrt{0.43858}} = 1.51$$

$$v_o = \frac{c}{n_o} = \frac{300000}{1.51} km/s = 198675.5\,\text{km/s}$$

Polarization (Fig 4) :

$$\boldsymbol{D}^{*(o)} = \left(\frac{m_2}{\sqrt{1-m_3^2}}\ ,\ \frac{-m_1}{\sqrt{1-m_3^2}}\ ,\ 0\right)$$

$$= \left(\frac{2}{\sqrt{5}}\ ,\ \frac{-1}{\sqrt{5}}\ ,\ 0\right)$$

$$= \left(25.57^o\ ,\ 116.57^o\ ,\ 90^o\right)$$

There is a coincidence between m and P : $\alpha_o = 0^o$

* For the extra-ordinary ray:

$$n_e = \frac{1}{\sqrt{m_3^2\eta_{11}^* + \left(1-m_3^2\right)\eta_{33}^*}} = \frac{1}{\sqrt{\frac{1}{6}\left(\eta_{11}^* + 5\eta_{33}^*\right)}} = 1.47646$$

$$v_e = \frac{c}{n_e} = \frac{300000}{1.47646} km/s = 203190.7\,\text{km/s}$$

Polarization (figure 4) :

$$\boldsymbol{D}^{*(e)} = \left(\frac{m_1 m_3}{\sqrt{1-m_3^2}}\ ,\ \frac{m_2 m_3}{\sqrt{1-m_3^2}}\ ,\ -\sqrt{1-m_3^2}\right)$$

$$= \left(\frac{1}{\sqrt{30}}\ ,\ \frac{2}{\sqrt{30}}\ ,\ \frac{-5}{\sqrt{30}}\right)$$

$$= \left(79.48^o\ ,\ 68.58^o\ ,\ 155.91^o\right)$$

Angle of lack of coincidence between m and P:

$$cos\alpha_e = \frac{m_3^2\eta_{11}^* + \left(1-m_3^2\right)\eta_{33}^*}{\sqrt{m_3^2\left(\eta_{11}^*\right)^2 + \left(1-m_3^2\right)\left(\eta_{33}^*\right)^2}}$$

$$= 0.9998069$$

$$\Rightarrow \alpha_e = 1.126^o$$

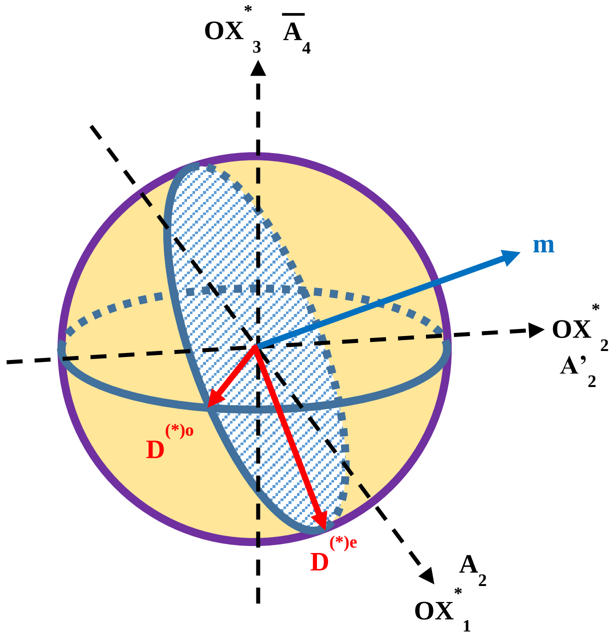


Fig 4. The polarization of the ordinary ray and extra-ordinary ray

CONCLUSION

Based on the general expression of refractive index (1), by the transformation cosinus matrix and tensorial calculations, we have completely solved the theme "Light propagation in optical uniaxial crystals". These analytical expressions describe all the properties of light propagating across the crystal. We have some remarks: the polarization of the two propagating modes depends only on the light direction whereas the light velocities and the angle of lack of coincidence between m and P of extra-ordinary ray depend on the crystal and light direction.

With the exception of the cubic system, which is an isotropic medium in optical aspect, our approach can be applied to orthorhombic and monoclinic systems. Of course the calculations will be more complex and take longer time because of in these cases $\eta^*_{11} \neq \eta^*_{22}$.

Acknowledgments: *The authors wish to give their thanks to Pr. Lê Khắc Bình for his comments and helps during the preparation of this article.*

Phương pháp giải tổng quát của chủ đề "Sự truyền ánh sáng trong các tinh thể đơn trục quang học"

- **Trương Quang Nghĩa**
- **Nguyễn Từ Ngọc Hương**

Trường Đại học Khoa học Tự nhiên, Đại học Quốc gia thành phố Hồ Chí Minh

TÓM TẮT

Trong bài báo này, chúng tôi giới thiệu một cách thức mới để nhận được phương pháp giải tổng quát có thể mô tả được tất cả các tính chất của sự truyền ánh sáng khi đi qua các tinh thể đơn trục quang học. Trong cách thức này, chúng tôi không sử dụng khái niệm chỉ số ellipsoid chiết suất như đã từng làm trong các tài liệu tham khảo. Phương pháp này đưa ra các biểu thức đại số nên chúng ta có thể dễ dàng tính toán hoặc viết một chương trình nhỏ để tính các biểu thức này.

Từ khóa*: tia bất thường, sự phân cực ánh sáng, vận tốc ánh sáng, hệ phương trình Maxwell, tinh thể đơn trục quang học, tia thường, chỉ số chiết suất, tensor*

REFERENCES

[1]. Q.H. Khang, Quang học tinh thể và kính hiển vi phân cực, Nxb. Đại học và Trung học chuyên nghiệp, Hà Nội (1986).

[2]. T.Q. Nghĩa, Tính chất vật lý của tinh thể, Nxb, Đại học Quốc gia TP. Hồ Chí Minh (2012).

[3]. N.V. Perelomova, M.M. Tagieva, *Problems in crystal physics*, Mir Publishers, Moscow (1983).