H^P estimation for the Cauchy problem for nonlinear elliptic equation

- **Le Duc Thang**

University of Science, VNU- HCM
Ho Chi Minh City Industry and Trade College

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ABSTRACT

In this paper, we investigate the Cauchy problem for a ND nonlinear elliptic equation in a bounded domain. As we know, the problem is severely ill-posed. We apply the Fourier truncation method to regularize the problem. Error estimates between the regularized solution and the exact solution are established in H^p space under some priori assumptions on the exact solution.

***Key words**: nonlinear elliptic equation, ill-posed problem, regularization, truncation method*

INTRODUCTION

In this paper, we consider the Cauchy problem for a nonlinear elliptic equation in a bounded domain. The problem has the form

$$\begin{cases} \Delta u = F(x', x_N, u(x', x_N)), & (x', x_N) \in \Omega \times (0,T), \\ u(x', x_N) = 0, & (x', x_N) \in \partial\Omega \times (0,T), \\ u(x', T) = \varphi(x'), & x' \in \Omega, \\ u_{x_N}(x', T) = 0, & x' \in \Omega. \end{cases} \tag{1}$$

Where T is a positive constant, $\Omega = (0,\pi)^{N-1}$, N is a natural number and $N \geq 2$, the function $\varphi \in L^2(\Omega)$ is known and F is called the source function. It is well-known the above problems is severely ill-posed in the sense of Hadamard. In fact, for a given final data, we are not sure that a solution of the problem exists. In the case a solution exists, it may not depend continuously on the final data. The problem has many various applications, for example in electrocardiography [7], astrophysics [6] and plasma physics [15, 16].

In the past, there have been many studies on the Cauchy problem for linear homogeneous elliptic equations, [1, 5, 9, 10, 12]. However, the literature on the nonlinear elliptic equation is quite scarce. We mention here a nonlinear elliptic problem of [13] with globally Lipschitz source terms, where authors approximated the problem by a truncation method. Using the method in [13,14], we study the Cauchy problem for nonlinear elliptic in multidimensional domain.

The paper is organized as follows. In Section 2, we present the solution of equation (1). In Section 3, we present the main results on regularization theory for local Lipschitz source function. We finish the paper with a remark.

SOLUTION OF THE PROBLEM

Assume that problem (1) has a unique solution $u(x', x_N)$. By using the method of separation of variables, we can show that solution of the problem has the form

$$u(x',x_N)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}...\sum_{n_{N-1}=1}^{\infty}\left[\cosh((T-x_N)\sqrt{n_1^2+n_2^2+...+n_{N-1}^2})\varphi_{n_1n_2...n_{N-1}}\right.$$

$$\left.+\int_{x_N}^{T}\frac{\sinh((\tau-x_N)\sqrt{n_1^2+n_2^2+...+n_{N-1}^2})}{\sqrt{n_1^2+n_2^2+...+n_{N-1}^2}}F_{n_1n_2...n_{N-1}}(u)(\tau)d\tau\right].\phi_{n_1n_2...n_{N-1}}(x'). \qquad (2)$$

Indeed, let $u(x',x_N)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}...\sum_{n_{N-1}=1}^{\infty}u_{n_1n_2...n_{N-1}}(x_N)\phi_{n_1n_2...n_{N-1}}(x')$ be the Fourier series in $L^2(\Omega)$ with orthonormal basis $\phi_{n_1n_2...n_{N-1}}(x')=\sqrt{\left(\frac{2}{\pi}\right)^{N-1}}\sin(n_1x_1)\sin(n_2x_2)...\sin(n_{N-1}x_{N-1})$. From (1), we can obtain the following ordinary differential equation

$$\begin{cases}\frac{d^2}{dx_N^2}u_{n_1n_2...n_{N-1}}(x_N)-(n_1^2+n_2^2+...+n_{N-1}^2)u_{n_1n_2...n_{N-1}}(x_N)=F_{n_1n_2...n_{N-1}}(u)(x_N), & x_N\in(0,T),\\ u_{n_1n_2...n_{N-1}}(T)=\varphi_{n_1n_2...n_{N-1}}, \\ \frac{d}{dx_N}u_{n_1n_2...n_{N-1}}(T)=0,\end{cases} \qquad (3)$$

where $F_{n_1n_2...n_{N-1}}(u)(x_N)=\int_{\Omega}F(x',x_N,u(x',x_N))\phi_{n_1n_2...n_{N-1}}dx'$, $\varphi_{n_1n_2...n_{N-1}}=\int_{\Omega}\varphi(x')\phi_{n_1n_2...n_{N-1}}(x')dx'$ and $u_{n_1n_2...n_{N-1}}=\int_{\Omega}u(x',x_N)\phi_{n_1n_2...n_{N-1}}(x')dx'$.

The equation (3) is ordinary differential equations. It is easy to see that its solution is given by

$$u_{n_1n_2...n_{N-1}}(x_N)=\cosh((T-x_N)\sqrt{n_1^2+n_2^2+...+n_{N-1}^2}\varphi_{n_1n_2...n_{N-1}}$$

$$+\int_{x_N}^{T}\frac{\sinh((\tau-x_N)\sqrt{n_1^2+n_2^2+...+n_{N-1}^2}}{\sqrt{n_1^2+n_2^2+...+n_{N-1}^2}}F_{n_1n_2...n_{N-1}}(u)(\tau)d\tau. \qquad (4)$$

REGULARIZATION AND ERROR ESTIMATE FOR NONLINEAR PROBLEM WITH LOCALLY LIPSCHITZ SOURCE

We know from (4) that, when $n_1,n_2,...,n_{N-1}$ become large , the terms $\cosh((T-x_N)\sqrt{n_1^2+n_2^2+...+n_{N-1}^2})$ and $\frac{\sinh((\tau-x_N)\sqrt{n_1^2+n_2^2+...+n_{N-1}^2})}{\sqrt{n_1^2+n_2^2+...+n_{N-1}^2}}$ increase rather quickly. Thus, these terms are the cause for instability. In this paper, we use the Fourier truncated method. The essence of the method is to eliminate all high frequencies from the solution, and consider the problem only for $n_1,n_2...n_{N-1}$ satisfying $\sqrt{n_1^2+n_2^2+...+n_{N-1}^2}\le C_\varepsilon$. Here C_ε is a constant which will be selected appropriately as a regularization parameter which satisfies $\lim_{\varepsilon\to 0}C_\varepsilon=+\infty$.

Let the function $F:\Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ such that: for each $M>0$ and for any u,v satisfying $|u|,\ |v| \le M$, there holds

$$\left|F(x',x_N,u)-F(x',x_N,v)\right| \le K_F(M)\left|u-v\right|, \tag{5}$$

where $(x',x_N)\in \Omega \times [0,T]$ and

$$K_F(M) := \sup\left\{\left|\frac{F(x',x_N,u)-F(x',x_N,v)}{u-v}\right| : \ |u|,\ |v| \le M,\ u\neq v,\ (x',x_N)\in\Omega \times [0,T]\right\} < +\infty \ .$$

We note that $K_F(M)$ is increasing and $\lim\limits_{M\to+\infty} K_F(M) = +\infty$. For all $M>0$, we approximate F by F_M defined by

$$F_M(x',x_N,u(x',x_N)) = \begin{cases} F(x',x_N,M), & u(x',x_N) > M, \\ F(x',x_N,u(x',x_N)), & -M \le u(x',x_N) \le M, \\ F(x',x_N,-M), & u(x',x_N) < -M. \end{cases}$$

For each $\varepsilon > 0$, we consider a parameter $M_\varepsilon \to +\infty$ as $\varepsilon \to 0$. We shall use the following well-posed problem

$$\begin{cases} \Delta v = P_{C_\varepsilon} F_{M_\varepsilon}\ x',x_N,v\ x',x_N\ \ , & x',x_N\ \in \Omega\times\ 0,T\ , \\ v\ x',x_N\ = 0, & x',x_N\ \in \partial\Omega\times\ 0,T\ , \\ v\ x',T\ = P_{C_\varepsilon}\varphi^\varepsilon\ x'\ ,\quad v_{x_N}\ x',T\ = 0, & x'\in\Omega. \end{cases} \tag{6}$$

where

$$P_{C_\varepsilon} w = \sum_{\substack{n_1,n_2\dots,n_{N-1}\ge 1 \\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\le C_\varepsilon}} \left\langle w,\phi_{n_1n_2\dots n_{N-1}}\right\rangle \phi_{n_1n_2\dots n_{N-1}} \qquad \text{for all } w\ \in L^2(\Omega).$$

We show that the solution $u_{\varepsilon,\varphi^\varepsilon}$ of problem (6) satisfies the following integral equation

$$\begin{aligned} u_{\varepsilon,\varphi^\varepsilon}(x',x_N) = \sum_{\substack{n_1,n_2\dots,n_{N-1}\ge 1 \\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\le C_\varepsilon}} & \Bigg[\cosh((T-x_N)\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2})\varphi^\varepsilon_{n_1n_2\dots n_{N-1}} + \\ & + \int_{x_N}^{T} \frac{\sinh((\tau-x_N)\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2})}{\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}}\ F_{M_\varepsilon\ \ n_1n_2\dots n_{N-1}}(u^\varepsilon)(\tau)d\tau\Bigg]\phi_{n_1n_2\dots n_{N-1}}(x'), \end{aligned} \tag{7}$$

Lemma 1. For $u_1(x',x_N),\ u_2(x',x_N)$, we have

$$\left|F_M(x',x_N,u_2(x',x_N)-F_M(x',x_N,u_1(x',x_N)\right| \le K_F(M)\left|u_2(x',x_N)-u_1(x',x_N)\right|.$$

Proof. If $u_1(x',x_N) < -M$ and $u_2(x',x_N) < -M$ then

$$\left|F_M(x',x_N,u_2(x',x_N)-F_M(x',x_N,u_1(x',x_N)\right| = 0 .$$

If $u_1(x',x_N) < -M \le u_2(x',x_N) \le M$ then

$$\begin{aligned} \left|F_M(x',x_N,u_2(x',x_N)-F_M(x',x_N,u_1(x',x_N))\right| &= \left|F_M(x',x_N,u_2(x',x_N)-F_M(x',x_N,-M)\right| \\ &\le K_F(M)\left|u_2(x',x_N)-u_1(x',x_N)\right|. \end{aligned}$$

If $u_1(x',x_N) < -M < M < u_2(x',x_N)$ then

$$\left|F_M(x',x_N,u_2(x',x_N)-F_M(x',x_N,u_1(x',x_N))\right|=\left|F_M(x',x_N,M)-F_M(x',x_N,-M)\right|$$
$$\le K_F(M)\left|u_2(x',x_N)-u_1(x',x_N)\right|.$$

If $-M\le u_1(x',x_N),\ u_2(x',x_N)\le M$ then

$$\left|F_M(x',x_N,u_2(x',x_N)-F_M(x',x_N,u_1(x',x_N))\right|=\left|F(x',x_N,u_2(x',x_N)-F(x',x_N,u_1(x',x_N))\right|$$
$$\le K_F(M)\left|u_2(x',x_N)-u_1(x',x_N)\right|.$$

This completes the proof.

Lemma 2. Let u be the exact solution to problem (1). Then we have the following estimate

$$\left\|u_{\varepsilon,\varphi^\varepsilon}(x_N)-P_{C_\varepsilon}u(x_N)\right\|_{L^2(\Omega)}\le 2\exp(2(T-x_N)C_\varepsilon)\left\|\varphi^\varepsilon-\varphi\right\|^2_{L^2(\Omega)}$$
$$+2K_F^2(M_\varepsilon)(T-x_N)\int_{x_N}^{T}\exp(2(\tau-x_N)C_\varepsilon)\left\|u_{\varepsilon,\varphi^\varepsilon}(\tau)-u(\tau)\right\|^2_{L^2(\Omega)}d\tau.$$

Proof. From the definition of $u_{\varepsilon,\varphi^\varepsilon}$ and u, we have

$$\left\|u_{\varepsilon,\varphi^\varepsilon}(x_N)-P_{C_\varepsilon}u(x_N)\right\|^2_{L^2(\Omega)}\le 2\sum_{\substack{n_1,n_2,\dots,n_{N-1}\ge 1\\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\le C_\varepsilon}}\left|\cosh((T-x_N)\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}(\varphi^\varepsilon_{n_1n_2\dots n_{N-1}}-\varphi_{n_1n_2\dots n_{N-1}})\right|^2$$

$$+2\sum_{\substack{n_1,n_2,\dots,n_{N-1}\ge 1\\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\le C_\varepsilon}}\left[\int_{x_N}^{T}\frac{\sinh((\tau-x_N)\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2})}{\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}}((F_{M_\varepsilon})_{n_1n_2\dots n_{N-1}}(u_{\varepsilon,\varphi^\varepsilon})(\tau)-F_{n_1n_2\dots n_{N-1}}(u)(\tau))d\tau\right]^2$$

$$\le 2\exp(2(T\text{-}x_N)C_\varepsilon)\left\|\varphi^\varepsilon-\varphi\right\|^2_{L^2(\Omega)}\ +2(T\text{-}x_N)\int_{x_N}^{T}\exp(2(\tau-x_N)C_\varepsilon\left\|F_{M_\varepsilon}(\tau,u_{\varepsilon,\varphi^\varepsilon}(\tau))-F(\tau,u(\tau))\right\|^2_{L^2(\Omega)}d\tau. \tag{8}$$

Since $\lim_{\varepsilon\to 0^+}M_\varepsilon=+\infty$, for a sufficiently small $\varepsilon>0$, there exists M_ε such that $M_\varepsilon\ge\left\|u\right\|_{L^\infty([0,T];L^2(\Omega))}$.

For M_ε we have $F_{M_\varepsilon}(x',x_N,u(x',x_N))=F(x',x_N,u(x',x_N))$. Using the Lipschitz property of F_M as in Lemma 1, we get

$$\left\|F_{M_\varepsilon}(\tau,u_{\varepsilon,\varphi^\varepsilon}(\tau))-F(\tau,u(\tau))\right\|^2_{L^2(\Omega)}\le K_F^2(M_\varepsilon)\left\|u_{\varepsilon,\varphi^\varepsilon}(\tau)-u(\tau)\right\|^2_{L^2(\Omega)}\ . \tag{9}$$

Combining (8) and (9), we complete the proof of Lemma 2. □

Theorem 1. Let $\varepsilon>0$ and let F be the function defined in (5). Then the problem (6) has a unique solution $u_{\varepsilon,\varphi^\varepsilon}\in C([0,T];L^2(\Omega))$.

Proof. We prove the equation (7) has a unique solution $u_{\varepsilon,\varphi^\varepsilon}\in C([0,T];L^2(\Omega))$. Put

$$\Phi(u_{\varepsilon,\varphi^\varepsilon})(x',x_N)=\psi(x',x_N)+G(x',x)$$

where

$$\psi(x',x_N)=\sum_{\substack{n_1,n_2,\dots,n_{N-1}\ge 1\\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\le C_\varepsilon}}\cosh((T-x_N)\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2})\varphi^\varepsilon_{n_1n_2\dots n_{N-1}}\phi_{n_1n_2\dots n_{N-1}}(x')$$

and

$$G(x',x_N)=\sum_{\substack{n_1,n_2,\dots,n_{N-1}\ge 1\\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\le C_\varepsilon}}\left(\int_{x_N}^{T}\frac{\sinh((\tau-x_N)\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2})}{\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}}\,F_{M_\varepsilon\,n_1n_2\dots n_{N-1}}(u_{\varepsilon,\varphi^\varepsilon})(\tau)d\tau\right)\phi_{n_1n_2\dots n_{N-1}}(x')$$

.

We claim that

$$\left\|\Phi^p(v_{\varepsilon,\varphi^\varepsilon})(x_N)-\Phi^p(w_{\varepsilon,\varphi^\varepsilon})(x_N)\right\|_{L^2(\Omega)}\le\sqrt{\frac{\left(K_F^2(M_\varepsilon)T\exp(2TC_\varepsilon)\right)^p}{p!}}\left\|\left|v_{\varepsilon,\varphi^\varepsilon}-w_{\varepsilon,\varphi^\varepsilon}\right|\right\| \tag{10}$$

for $p\ge 1$, where $\||\cdot|\|$ is the sup norm in $C([0,T];L^2(\Omega))$. We shall prove the above inequality by induction.

For $p=1$, using the inequality

$$\int_{x_N}^{T}\frac{\sinh((\tau-x_N)\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2})}{\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}}d\le\exp(2\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}T)T$$

and using Lemma 1, we have

$$\left\|\Phi(v_{\varepsilon,\varphi^\varepsilon})(x_N)-\Phi(w_{\varepsilon,\varphi^\varepsilon})(x_N)\right\|_{L^2(\Omega)}^2=$$

$$\sum_{\substack{n_1,n_2,\dots,n_{N-1}\ge 1\\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\le C_\varepsilon}}\left[\int_{x_N}^{T}\frac{\sinh((\tau-x_N)\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2})}{\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}}\left(F_{M_\varepsilon\,n_1n_2\dots n_{N-1}}(v_{\varepsilon,\varphi^\varepsilon})(\tau)-F_{M_\varepsilon\,n_1n_2\dots n_{N-1}}(w_{\varepsilon,\varphi^\varepsilon})(\tau)\right)d\tau\right]^2$$

$$\le\exp(2TC_\varepsilon)T\int_{x_N}^{T}\left[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\dots\sum_{n_{N-1}=1}^{\infty}\left|F_{M_\varepsilon\,n_1n_2\dots n_{N-1}}(v_{\varepsilon,\varphi^\varepsilon})(\tau)-F_{M_\varepsilon\,n_1n_2\dots n_{N-1}}(w_{\varepsilon,\varphi^\varepsilon})(\tau)\right|^2\right]d\tau$$

$$\le\exp(2TC_\varepsilon)T\int_{x_N}^{T}\left\|F_{M_\varepsilon}(\tau,v_{\varepsilon,\varphi^\varepsilon}(\tau))-F_{M_\varepsilon}(\tau,w_{\varepsilon,\varphi^\varepsilon}(\tau))\right\|_{L^2(\Omega)}^2d\tau\le K_F^2(M_\varepsilon)\exp(2TC_\varepsilon)T^2\left\|\left|v_{\varepsilon,\varphi^\varepsilon}-w_{\varepsilon,\varphi^\varepsilon}\right|\right\|^2.$$

Thus (10) holds for $p=1$. Suppose that (10) holds for $p=k$. We prove that (10) holds for $p=k+1$.

We have

$$\left\|\Phi^{k+1}(v_{\varepsilon,\varphi^\varepsilon})(x_N)-\Phi^{k+1}(w_{\varepsilon,\varphi^\varepsilon})(x_N)\right\|_{L^2(\Omega)}^2\le\exp(2TC_\varepsilon)T\int_{x_N}^{T}\left\|F_{M_\varepsilon}(\tau,\Phi^k(v_{\varepsilon,\varphi^\varepsilon}(\tau)))-F_{M_\varepsilon}(\tau,\Phi^k(w_{\varepsilon,\varphi^\varepsilon}(\tau)))\right\|_{L^2(\Omega)}^2d\tau$$

$$\le K_F^2(M_\varepsilon)\exp(2TC_\varepsilon)T\int_{x_N}^{T}\left\|\Phi^k(v_{\varepsilon,\varphi^\varepsilon})(\tau)-\Phi^k(w_{\varepsilon,\varphi^\varepsilon})(\tau)\right\|_{L^2(\Omega)}^2d\tau$$

$$\left\|\Phi^{k+1}(v_{\varepsilon,\varphi^\varepsilon})(x_N)-\Phi^{k+1}(w_{\varepsilon,\varphi^\varepsilon})(x_N)\right\|_{L^2(\Omega)}^2\le K_F^2(M_\varepsilon)K_F^{2k}\exp(2TC_\varepsilon)T\exp(2TC_\varepsilon k)\frac{\left(T-x_N\right)^{k+1}}{(k+1)!}\left\|\left|v_{\varepsilon,\varphi^\varepsilon}-w_{\varepsilon,\varphi^\varepsilon}\right|\right\|^2.$$

Therefore, we get

$$\left\|\Phi^p(v_{\varepsilon,\varphi^\varepsilon})(x_N)-\Phi^p(w_{\varepsilon,\varphi^\varepsilon})(x_N)\right\|_{L^2(\Omega)} \le \sqrt{\frac{K_F^2(M_\varepsilon)T\exp(2TC_\varepsilon)^{\ p}}{p!}}\left\|v_{\varepsilon,\varphi^\varepsilon}-w_{\varepsilon,\varphi^\varepsilon}\right\| , \tag{11}$$

for all $v_{\varepsilon,\varphi^\varepsilon}, w_{\varepsilon,\varphi^\varepsilon} \in C([0,T];L^2(\Omega))$.

Let us consider $\Phi : C([0,T];L^2(\Omega)) \to C([0,T];L^2(\Omega))$. It is easy to see that

$$\lim_{p\to+\infty}\sqrt{\frac{K_F^2(M_\varepsilon)T\exp(2TC_\varepsilon)^{\ p}}{p!}} = 0.$$

As a consequence, there exists a positive integer number p_0 such that Φ^{p_0} is a contraction. It follows that the equation $\Phi^{p_0}(u) = u$ has a unique solution $u_{\varepsilon,\varphi^\varepsilon} \in C([0,T];L^2(\Omega))$. We claim that $\Phi(u_{\varepsilon,\varphi^\varepsilon}) = u_{\varepsilon,\varphi^\varepsilon}$.
In fact, one has $\Phi(\Phi^{p_0}(u_{\varepsilon,\varphi^\varepsilon})) = \Phi(u_{\varepsilon,\varphi^\varepsilon})$. By the uniqueness of the fixed point of Φ^{p_0} , one has $\Phi(u_{\varepsilon,\varphi^\varepsilon}) = u_{\varepsilon,\varphi^\varepsilon}$, i.e., the equation $\Phi(u) = u$ has a unique solution $u_{\varepsilon,\varphi^\varepsilon} \in C([0,T];L^2(\Omega))$.

To show error estimates between the exact solution and the regularized solution, we need the exact solution belonging to the Gevrey space.

Definition 1. (Gevrey-type space). (see [2, 3]) The Gevrey class of functions of order $s > 0$ and index $\sigma > 0$ is denoted by $G_\sigma^{s/2}$ and is defined as

$$G_\sigma^{s/2} = \left\{ f \in L^2(\Omega) : \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\dots\sum_{n_{N-1}=1}^{\infty}(n_1^2+n_2^2+\dots+n_{N-1}^2)^{s/2}\exp(2\sigma\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2})\,|\left\langle f,\phi_{n_1n_2\dots n_{N-1}}\right\rangle|^2 < \infty \right\}.$$

It is a Hilbert space with the following norm

$$\|f\|_{G_\sigma^{s/2}} = \sqrt{\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\dots\sum_{n_{N-1}=1}^{\infty}(n_1^2+n_2^2+\dots+n_{N-1}^2)^{s/2}\exp(2\sigma\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2})\,|\left\langle f,\phi_{n_1n_2\dots n_{N-1}}\right\rangle|^2}.$$

For a Hilbert space H, we denote $L^\infty(0,T;H) = \left\{ f:[0,T]\to H \mid ess\sup_{0\le t\le T}|f(t)|_H < \infty \right\}$

and

$$\|f\|_{L^\infty(0,T;H)} = ess\sup_{0\le t\le T}|f(t)|_H .$$

We consider some assumptions on the exact solution as the following:

$$ess\sup_{0\le x_N\le T}\sqrt{\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\dots\sum_{n_{N-1}=1}^{\infty}(n_1^2+n_2^2+\dots+n_{N-1}^2)^{\beta}\exp\left(2x_N\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\right)u_{n_1n_2\dots n_{N-1}}^2(x_N)} \le I_1, \tag{12}$$

$$ess\sup_{0\le x_N\le}\sqrt{\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\dots\sum_{n_{N-1}=1}^{\infty}\exp(2(x_N+\alpha)\sqrt{n_1^2+n_2^2+\dots n_{N-1}^2})u_{n_1n_2\dots n_{N-1}}^2(x_N)} \le I_2,$$

(13)

for all $x_N \in [0,T]$, where $\alpha,\ \beta,\ I_1,\ I_2,$ are positive constants.

Lemma 3. For any $w \in G_\sigma^k$, we have the following inequality

$$\left\|w - P_{C_\varepsilon}w\right\|_{L^2(\Omega)} \le C_\varepsilon^{-k}e^{-\sigma C_\varepsilon}\|w\|_{G_\sigma^k}.$$

Proof. For $w \in G_\sigma^k$, we get

$$\left\| w - P_{C_\varepsilon} w \right\|_{L^2(\Omega)}^2 = \sum_{\substack{n_1,n_2,\dots,n_{N-1}\geq 1 \\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}>C_\varepsilon}} \left| \left\langle w, \phi_{n_1 n_2 \dots n_{N-1}} \right\rangle \right|^2$$

$$\leq C_\varepsilon^{-2k} e^{-2\sigma C_\varepsilon} \sum_{\substack{n_1,n_2,\dots,n_{N-1}\geq 1 \\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}>C_\varepsilon}} \left(n_1^2 + n_2^2 + \dots + n_{N-1}^2 \right)^k \exp\left(2\sigma \sqrt{n_1^2 + n_2^2 + \dots + n_{N-1}^2} \right) \left| \left\langle w, \phi_{n_1 n_2 \dots n_{N-1}} \right\rangle \right|^2$$

$$\leq C_\varepsilon^{-2k} e^{-2\sigma C_\varepsilon} \left\| w \right\|_{G_\sigma^k}.$$

This completes the proof.

The following theorem provides some error estimates in the $L^2 - norm$ when the exact solution belongs to the Gevrey space.

Theorem 2. Assume that the problem (1) has a unique solution u which satisfies (12). If C_ε and M_ε are chosen such that $\lim_{\varepsilon\to 0} \varepsilon e^{TC_\varepsilon} = 0$ and $\lim_{\varepsilon\to 0} \exp(2K_F^2(M_\varepsilon)T^2) C_\varepsilon^{-\beta} = \lim_{\varepsilon\to 0} \exp(2K_F^2(M_\varepsilon)T^2)\varepsilon e^{TC_\varepsilon} = 0$, then we have

$$\left\| u_{\varepsilon,\varphi^\varepsilon}(x_N) - u(x_N) \right\|_{L^2(\Omega)} \leq \sqrt{2C_\varepsilon^{-2\beta} I_2^2 + 4\varepsilon^2 e^{2TC_\varepsilon}} \exp(2K_F^2(M_\varepsilon)T^2) e^{-x_N C_\varepsilon}. \tag{14}$$

Proof. Since $u \in G_{x_N}^\beta$ then using Lemma 3, we get

$$\left\| u(x_N) - P_{C_\varepsilon} u(x_N) \right\|_{L^2(\Omega)}^2 \leq C_\varepsilon^{-2\beta} e^{-2x_N C_\varepsilon} \left\| w \right\|_{G_{x_N}^\beta}^2.$$

Lemma 2 and the triangle inequality lead to

$$\left\| u_{\varepsilon,\varphi^\varepsilon}(x_N) - u(x_N) \right\|_{L^2(\Omega)}^2 \leq 2 \left\| u_{\varepsilon,\varphi^\varepsilon}(x_N) - P_{C_\varepsilon} u(x_N) \right\|_{L^2(\Omega)}^2 + 2 \left\| u(x_N) - P_{C_\varepsilon} u(x_N) \right\|_{L^2(\Omega)}^2$$

$$\leq 2\mathrm{C}_\varepsilon^{-2\beta} e^{-2x_N C_\varepsilon} \left\| u(x_N) \right\|_{G_{x_N}^\beta}^2 + 4\exp(2(T - x_N)C_\varepsilon) \left\| \varphi^\varepsilon - \varphi \right\|_{L^2(\Omega)}^2$$

$$+4K_F^2(M_\varepsilon)(T - x_N) \int_{x_N}^T \exp(2(\tau - x_N)C_\varepsilon) \left\| u_{\varepsilon,\varphi^\varepsilon}(\tau) - u(\tau) \right\|_{L^2(\Omega)}^2 d\tau. \tag{15}$$

Multiplying (15) by $e^{2x_N C_\varepsilon}$ and applying Gronwall's inequality, we get

$$e^{2x_N C_\varepsilon} \left\| u_{\varepsilon,\varphi^\varepsilon}(x_N) - u(x_N) \right\|_{L^2(\Omega)}^2 \leq \left[2C_\varepsilon^{-2\beta} \sup_{0\leq x_N \leq T} \left\| u(x_N) \right\|_{G_{x_N}^\beta}^2 + 4e^{2TC_\varepsilon}\varepsilon^2 \right] \exp\left(4K_F^2(M_\varepsilon)T^2 \right),$$

which leads to the desired result

$$\left\| u_{\varepsilon,\varphi^\varepsilon}(x_N) - u(x_N) \right\|_{L^2(\Omega)}^2 \leq \sqrt{2C_\varepsilon^{-2\beta} I_2^2 + 4e^{2TC_\varepsilon}\varepsilon^2} \exp\left(2K_F^2(M_\varepsilon)T^2 \right) e^{-x_N C_\varepsilon}.$$

This completes the proof.

The next theorem provides an error estimate in the Hilbert scales $\{H^p(\Omega)\}_{p\in\mathbb{R}}$ which is equipped with a norm defined by

$$\|f\|^2_{H^p(\Omega)} = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\dots\sum_{n_{N-1}=1}^{\infty}(n_1^2+n_2^2+\dots+n_{N-1}^2)^p\left|\left\langle f,\phi_{n_1n_2\dots n_{N-1}}\right\rangle\right|^2.$$

Theorem 3. Assume that the problem (1) has a unique u which satisfies (13). Let us choose C_ε and M_ε such that $\lim_{\varepsilon\to 0}\varepsilon C_\varepsilon^p e^{TC_\varepsilon}=0$ and $\lim_{\varepsilon\to 0}\mathrm{e}^{K_F^2(M_\varepsilon)T^2}e^{-\alpha C_\varepsilon}C_\varepsilon^p=\lim_{\varepsilon\to 0}\mathrm{e}^{K_F^2(M_\varepsilon)T^2}\varepsilon C_\varepsilon^p e^{TC_\varepsilon}=0$, then we have

$$\left\|u_{\varepsilon,\varphi^\varepsilon}(x_N)-u(x_N)\right\|_{H^p(\Omega)}\le\left[(\sqrt{2}+1)e^{K_F^2(M_\varepsilon)T^2}e^{-\alpha C_\varepsilon}I_3+2e^{K_F^2(M_\varepsilon)T^2}\varepsilon e^{TC_\varepsilon}\right]C_\varepsilon^p e^{-x_NC_\varepsilon},\qquad x_N\in[0,T].$$

Proof. First, we have

$$\left\|u_{\varepsilon,\varphi^\varepsilon}(x_N)-P_{C_\varepsilon}u(x_N)\right\|^2_{H^p(\Omega)}=\sum_{\substack{n_1,n_2,\dots,n_{N-1}\ge 1\\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\le C_\varepsilon}}\left(n_1^2+n_2^2+\dots+n_{N-1}^2\right)^p\left|u_{\varepsilon,\varphi^\varepsilon}(x',x_N)-u(x',x_N),\phi_{n_1n_2\dots n_{N-1}}(x')\right|^2$$

$$\left\|u_{\varepsilon,\varphi^\varepsilon}(x_N)-P_{C_\varepsilon}u(x_N)\right\|^2_{H^p(\Omega)}\le C_\varepsilon^{2p}\sum_{\substack{n_1,n_2,\dots,n_{N-1}\ge 1\\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\le C_\varepsilon}}\left|u_{\varepsilon,\varphi^\varepsilon}(x',x_N)-u(x',x_N),\phi_{n_1n_2\dots n_{N-1}}(x')\right|^2\le C_\varepsilon^{2p}\left\|u_{\varepsilon,\varphi^\varepsilon}(x_N)-u(x_N)\right\|^2_{L^2(\Omega)}.$$

It follows from theorem 2 that

$$\left\|u_{\varepsilon,\varphi^\varepsilon}(x_N)-P_{C_\varepsilon}u(x_N)\right\|_{H^p(\Omega)}\le\exp(2K_F^2(M_\varepsilon)T^2)C_\varepsilon^p e^{-x_NC_\varepsilon}\sqrt{2e^{-2\alpha C_\varepsilon}\sup_{0\le x_N\le T}\left\|u(x_N)\right\|^2_{G^0_{x_N+\alpha}}+4\varepsilon^2e^{2TC_\varepsilon}}\quad(16)$$

On the other hand, we consider the function

$$G(\xi)=\xi^pe^{-D\xi},\qquad D>0.$$

Since $G'(\xi)=\xi^{p-1}e^{-D\xi}(p-D\xi)$, it follows that G is decreasing when $D\xi\ge p$. Thus if $\varepsilon\le e^{\frac{-p(T+\alpha)}{2\alpha}}$ i.e., $2(x_N+\alpha)C_\varepsilon\ge p$, then for $n_1^2+n_2^2+\dots+n_{N-1}^2\ge C_\varepsilon^2$, we get

$$\left(n_1^2+n_2^2+\dots+n_{N-1}^2\right)^p\exp\left(-2(x_N+\alpha)\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\right)\le C_\varepsilon^{2p}e^{-2(x_N+\alpha)C_\varepsilon},$$

and

$$\begin{aligned}&\left\|u(x_N)-P_{C_\varepsilon}u(x_N)\right\|^2_{H^p(\Omega)}\\&=\sum_{\substack{n_1,n_2,\dots,n_{N-1}\ge 1\\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\ge C_\varepsilon}}\left(n_1^2+n_2^2+\dots+n_{N-1}^2\right)^p\left|\left\langle u(x',x_N),\phi_{n_1n_2\dots n_{N-1}}(x')\right\rangle\right|^2\\&\le C_\varepsilon^{2p}\exp\left(-2(x_N+\alpha)C_\varepsilon\right)\sum_{\substack{n_1,n_2,\dots,n_{N-1}\ge 1\\ \sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\ge C_\varepsilon}}\exp\left(2(x_N+\alpha)\sqrt{n_1^2+n_2^2+\dots+n_{N-1}^2}\right)\left|\left\langle u(x',x_N),\phi_{n_1n_2\dots n_{N-1}}(x')\right\rangle\right|^2\\&\le C_\varepsilon^{2p}e^{-(x_N+\alpha)C_\varepsilon}\sup_{0\le x_N\le T}\left\|u(x_N)\right\|_{G^0_{x_N+\alpha}}.\end{aligned}$$

Therefore

$$\left\|u(x_N)-P_{C_\varepsilon}u(x_N)\right\|_{H^p(\Omega)}\le C_\varepsilon^pe^{-(x_N+\alpha)C_\varepsilon}\sup_{0\le x_N\le T}\left\|u(x_N)\right\|_{G^0_{x_N+\alpha}}\qquad(17)$$

Combining (16) and (17), we get

$$\left\| u_{\varepsilon,\varphi^\varepsilon}(x_N) - u(x_N) \right\|_{H^p(\Omega)} \le \left\| u_{\varepsilon,\varphi^\varepsilon}(x_N) - P_{C_\varepsilon} u(x_N) \right\|_{H^p(\Omega)} + \left\| u(x_N) - P_{C_\varepsilon} u(x_N) \right\|_{H^p(\Omega)}$$

$$\le \left[\exp\ 2K_F^2(M_\varepsilon)T^2 \ \sqrt{2e^{-2\alpha C_\varepsilon} \sup_{0 \le x_N \le T} \left\| u(x_N) \right\|^2_{G^0_{x_N+\alpha}} + 4\varepsilon^2 e^{2TC_\varepsilon}} + \sup_{0 \le x_N \le T} \left\| u(x_N) \right\|_{G^0_{x_N+\alpha}} e^{-\alpha C_\varepsilon} \right] C_\varepsilon^p e^{-x_N C_\varepsilon}.$$

The inequality $\sqrt{a^2+b^2} \le a+b$ for $a,\ b \ge 0$ leads to

$$\left\| u_{\varepsilon,\varphi^\varepsilon}(x_N) - u(x_N) \right\|_{H^p(\Omega)} \le \left[(\sqrt{2}+1) \mathrm{e}^{K_F^2(M_\varepsilon)T^2} e^{-\alpha C_\varepsilon} I_3 + 2e^{K_F^2(M_\varepsilon)T^2} \varepsilon e^{TC_\varepsilon} \right] C_\varepsilon^p e^{-x_N C_\varepsilon}. \ \square$$

Remark 1. In theorem 2, let us choose $C_\varepsilon = \frac{\gamma}{T} \ln\left(\frac{1}{\varepsilon}\right)$, for $\gamma \in (0,1)$ and M_ε such that

$$K_F\ M_\varepsilon = \frac{1}{\sqrt{2}T} \sqrt{\beta - r\ \ln\left(\frac{\gamma}{T}\ln\left(\frac{1}{\varepsilon}\right)\right)},$$

for $r \in\ 0,\beta$. It is easy to check that $\lim_{\varepsilon \to 0} \exp(2K_F^2(M_\varepsilon)T^2) C_\varepsilon^{-\beta} = \lim_{\varepsilon \to 0} \exp(2K_F^2(M_\varepsilon)T^2) \varepsilon e^{TC_\varepsilon} = 0$.

Then (14) becomes

$$\left\| u_{\varepsilon,\varphi^\varepsilon}(x_N) - u(x_N) \right\|_{L^2(\Omega)} \le \sqrt{2I_2^2 + 4\varepsilon^{2-2\gamma} \left[\frac{\gamma}{T}\ln\left(\frac{1}{\varepsilon}\right)\right]^{2\beta}} \left[\frac{\gamma}{T}\ln\left(\frac{1}{\varepsilon}\right)\right]^{-r} \varepsilon^{\frac{x_N \gamma}{T}}.$$

CONCLUSION

In this paper, we investigate the Cauchy problem for a ND nonlinear elliptic equation in a bounded domain. We apply the Fourier truncation method for regularizing the problem. Error estimates between the regularized solution and exact solution are established in H^P space under some priori assumptions on the exact solution. In future, we will consider the Cauchy problem for a coupled system for nonlinear elliptic equations in three dimensions.

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Đánh giá H^P cho bài toán Cauchy cho phương trình elliptic phi tuyến

- **Lê Đức Thắng**

Trường Đại học Khoa học Tự nhiên, ĐHQG-HCM
Trường Cao Đẳng Công Thương TPHCM

TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu bài toán Cauchy cho phương trình elliptic phi tuyến trên miền bị chặn trong không gian nhiều chiều. Như đã biết, bài toán này là không chỉnh. Chúng tôi sử dụng phương pháp chặt cụt Fourier để chỉnh hóa nghiệm của bài toán. Đánh giá sai số giữa nghiệm chỉnh hóa và nghiệm chính xác được thiết lập trong không gian H^P với các giả thiết cho trước về tính trơn của nghiệm chính xác.

Từ khóa: *phương trình elliptic phi tuyến, bài toán không chỉnh, chỉnh hóa, phương pháp chặt cụt*

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