Regularization of a Cauchy problem for the heat equation

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ABSTRACT

In this paper, we study a Cauchy problem for the heat equation with linear source in the form

\[
\begin{align*}
\frac{\partial u}{\partial t}(x,t) &= \frac{\partial^2 u}{\partial x^2}(x,t) + f(x,t), \\
\frac{\partial u}{\partial x}(x,0) &= \phi_0(x), \\
\frac{\partial u}{\partial x}(L,t) &= \psi_0(t), \\
\frac{\partial u}{\partial x}(0,t) &= \phi_0(t), \\
\end{align*}
\]

This problem is ill-posed in the sense of Hadamard. To regularize the problem, the truncation method is proposed to solve the problem in the presence of noisy Cauchy data \(\epsilon\phi\) and \(\epsilon\psi\) satisfying

\[
\|\phi - \phi_0\| + \|\psi - \psi_0\| \leq \epsilon
\]

and that \(f\) satisfying \(\|f - f_0\| \leq \epsilon\). We give some error estimates between the regularized solution and the exact solution under some different a-priori conditions of exact solution.

Key words: elliptic equation, ill-posed problem, cauchy problem, regularization method, truncation method

INTRODUCTION

In this paper, the temperature \(u(x,t)\) for \((x,t) \in [0,L] \times [0,2\pi]\) is sought from known boundary temperature \(u(L,t) = \phi(t)\) and heat flux \(u_x(L,t) = \psi(t)\) measurements satisfying the following problem:

\[
\begin{align*}
\frac{\partial u}{\partial t}(x,t) &= u_{xx}(x,t) + f(x,t), \\
\frac{\partial u}{\partial x}(x,0) &= \phi_0(x), \\
\frac{\partial u}{\partial x}(L,t) &= \psi_0(t), \\
\frac{\partial u}{\partial x}(0,t) &= \phi_0(t), \\
\end{align*}
\]

where \(\phi_0, \psi_0\) are given functions (usually in \(L^2(0,2\pi)\)) and \(f\) is a given linear heat source which may depend on the independent variables \((x,t)\).

Note that we have no initial condition prescribed at \(t = 0\) and moreover, the Cauchy data \(\phi\) and \(\psi\) are perturbed so as to contain measurement errors in the form of the input noisy Cauchy data \(\phi_\epsilon\) and \(\psi_\epsilon\) (also in \(L^2(0,2\pi)\)) satisfying

\[
\begin{align*}
\|\phi - \phi_\epsilon\| + \|\psi - \psi_\epsilon\| &\leq \epsilon, \\
\end{align*}
\]

where \(\|\cdot\|\) denotes the \(L^2(0,2\pi)\)-norm and \(\epsilon > 0\) is a small positive number representing the level of noise.

It is well-known that, at least in the linear case, the problem (1) has at most one solution using classical analytical sideways continuation for the parabolic heat equation. The existence of solution also holds, in the case \(f = 0\). However, the problem is still ill-posed in the sense that the solution, if it exists, does not depend continuously on the data. Any small perturbation in the observation data can cause large errors in the solution \(u(x,t)\) for \(x \in [0,L]\). Therefore, most classical numerical methods often fail to give an acceptable approximation of the solution. Thus regularization techniques are required to stabilize the solution [3].

In recent years, the homogeneous sideways heat equation, i.e., \(f = 0\) in the first equation in (1), has been researched by many authors and various methods have been proposed, e.g. the difference regularization method [8], the boundary element Tikhonov regularization method [5], the Fourier method [9], the quasi-reversibility method [1, 6], the wavelet, wavelet-
Galérkin and spectral regularization methods [2, 7], the conjugate gradient method [4], to mention only a few.

To the best of our knowledge, the Cauchy problem for the linear sideways heat equation has not yet been. Therefore, in the present paper, we propose a new method that is based on linear integral equation to regularize problem (1) under two a priori conditions on the exact solution.

As will be shown in next section, for the linear sideways heat problem (1), its solution (exact solution) can be represented as an integral equation which contains some instability terms. In order to restore the stability we replace these instable terms by some regularization ones and show that the solution of our regularized problem converges to the solution of the original linear problem (if such solution exists), as the regularization parameter tends to zero. In the non-homogeneous problem, we have many choises of stability terms for regularization. However, in the case of non-homogeneous problem, the main solution \( u \) is complicated and is defined by a linear integral equation whose the right-hand side depends on the independent variables \((x, t)\). In this paper, we develop a truncation method to solve in a stable manner this linear integral equation.

**THE MAIN RESULTS**

Let \( \langle \cdot \rangle \) denote the inner product in \( L^2(0, 2\pi) \), and \( \varepsilon > 0 \) represent the noise level in 2. For \( \xi \in L^2(0, 2\pi) \), we have the Fourier series \( \xi(t) = \sum_{n \in \mathbb{Z}} \langle \xi(t), \exp(-int) \rangle \exp(int) \), where \( \langle \xi(t), \exp(-int) \rangle = \frac{1}{2\pi} \int_0^{2\pi} \xi(t) \exp(-int) \, dt \). The \( L^2(0, 2\pi) \)-norm of \( \xi \) is

\[
\|\xi\|^2 = 2\pi \sum_{n \in \mathbb{Z}} \left| \langle \xi(t), \exp(-int) \rangle \right|^2.
\]  
(3)

The principal value of \( \sqrt{n} \) is

\[
\sqrt{\text{in}} = \begin{cases} 
(1+i) \sqrt{\ln |n| / 2} , & n \geq 0, \\
(1-i) \sqrt{\ln |n| / 2} , & n < 0.
\end{cases}
\]  
(4)

Suppose that the solution of problem (1) is represented as a Fourier series

\[
u(x, t) = \sum_{n \in \mathbb{Z}} u_n(x) \exp(int), \quad \text{with} \quad u_n(x) = \langle u(x, t), \exp(-int) \rangle = \frac{1}{2\pi} \int_0^{2\pi} u(x, t) \exp(-int) \, dt.
\]  
From (1), we have the following systems of second-order ordinary differential equations:

\[
\begin{aligned}
\frac{d^2 u_n}{dx^2}(x) + i n u_n(x) &= f_n(x), & 0 < x < L, \\
u_n(L) &= \varphi_n = \langle \varphi(t), \exp(-int) \rangle, & t \in (0, 2\pi), \\
\frac{d u_n}{dx}(L) &= \psi_n = \langle \psi(t), \exp(-int) \rangle, & t \in (0, 2\pi),
\end{aligned}
\]  
(5)

where \( f_n(x) = \langle f(x, t), \exp(-int) \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x, t) \exp(-int) \, dt \) for all \( n \in \mathbb{Z} \).
For $n \in \mathbb{Z} \setminus \{0\}$, multiplying the first equation in (5) by $\frac{\sinh((\tau-x)\sqrt{ln})}{\sqrt{ln}}$ and integrating both sides from $x$ to $L$, we obtain

$$u_n(x) = \cosh((L-x)\sqrt{ln})u_n(L) - \frac{\sinh((L-x)\sqrt{ln})}{\sqrt{ln}}u'_n(L) - \int_x^L \frac{\sinh((\tau-x)\sqrt{ln})}{\sqrt{ln}}f_n(\tau)\,d\tau, \quad n \in \mathbb{Z} \setminus \{0\}. \tag{6}$$

In the case $n = 0$, multiplying the first equation in (5) by $(\tau-x)$ and integrating both sides from $x$ to $L$, we obtain

$$u_0(x) = u_0(L) - (L-x)u'_0(L) - \int_x^L (\tau-x)f_0(\tau)\,d\tau. \tag{7}$$

From (6) - (7) the exact form of $u$ is given by

$$u(x,t) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \left[ \cosh((L-x)\sqrt{ln})\varphi_n - \frac{\sinh((L-x)\sqrt{ln})}{\sqrt{ln}}\varphi_n - \int_x^L \frac{\sinh((\tau-x)\sqrt{ln})}{\sqrt{ln}}f_n(\tau)d\tau \right] \exp(int)$$

$$+ \Theta(\varphi_0, \varphi_0, f_0)(x), \tag{8}$$

where $\Theta(\varphi_0, \varphi_0, f_0)(x) = \varphi_0 - (L-x)\varphi'_0 - \int_x^L (\tau-x)f_0(\tau)d\tau$. In a few sentences, we present a brief introduction

Fourier truncated method. From equation (8), it can be observed that $\cosh((L-x)\sqrt{ln})$, $\frac{\sinh((L-x)\sqrt{ln})}{\sqrt{ln}}$ and $\frac{\sinh((\tau-x)\sqrt{ln})}{\sqrt{ln}}$ are unbounded, as $n$ tends to infinity, so in order to guarantee the convergence of the solution $u$ given by (8), the coefficient $(\varphi_n, \psi_n)$ must decay rapidly. But such a decay usually cannot occur for the measured data $(\varphi_n, \psi_n)$. Hence, a natural way is to eliminate the high frequencies and consider the solution $u$ for $n \leq N_\varepsilon$, where $N_\varepsilon$ is a positive integer; this is the so-called Fourier truncated method, and $N_\varepsilon$ plays the role of a regularization parameter satisfying $\lim_{\varepsilon \to 0} N_\varepsilon = +\infty$. We define the following two operators:

$$Q_{N_\varepsilon}^+(\varphi, \psi, f)(x,t) = \sum_{k \in \mathbb{Z} \setminus \{0\}} Q_{N_\varepsilon}^+(\varphi, \psi, f)(x)\exp(int)$$

$$= \frac{1}{2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \left[ \exp((-L-x)\sqrt{ln})\varphi_n + \frac{\exp(-L-x)\sqrt{ln}}{\sqrt{ln}}\psi_n + \int_x^L \frac{\exp(-\tau-x)\sqrt{ln}}{\sqrt{ln}}f_n(\tau)d\tau \right] \exp(int), \tag{9}$$

$$Q_{N_\varepsilon}^-(\varphi, \psi, f)(x,t) = \sum_{k \in \mathbb{Z} \setminus \{0\}} Q_{N_\varepsilon}^-(\varphi, \psi, f)(x)\exp(int)$$

$$= \frac{1}{2} \sum_{k \in \mathbb{Z} \setminus \{0\}} \left[ \exp((L-x)\sqrt{ln})\varphi_n + \frac{\exp(L-x)\sqrt{ln}}{\sqrt{ln}}\psi_n + \int_x^L \frac{\exp(\tau-x)\sqrt{ln}}{\sqrt{ln}}f_n(\tau)d\tau \right] \exp(int). \tag{10}$$

To approximate $u$, we introduce the regularized solution

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\[ u_{n}^{'}(x,t) = \sum_{k=N_{n},n>0} \left[ \cosh \left( (L-x)\sqrt{\ln} \right) \phi_{n}^{\prime} - \frac{\sinh \left( (L-x)\sqrt{\ln} \right)}{\sqrt{\ln}} \psi_{n}^{\prime} - \frac{i}{\sqrt{\ln}} \int_{x}^{(\tau-x)\sqrt{\ln}} f_{n}^{\prime}(\tau) \, d\tau \right] \exp(int) \]
\[ + Q_{n}(\phi^{\prime},\psi^{\prime},f^{\prime})(x,t) + \Theta(\phi_{n}^{\prime},\psi_{n}^{\prime},f_{n}^{\prime})(x). \]

(11)

Our these results would be applied after any necessary minor modifications have been made.

Lemma 1. For \( n \in \mathbb{Z} \setminus \{0\} \) and \( |n| \leq M \), we have the following inequalities:

\[ |\cosh (L-x)\sqrt{\ln}| \leq \exp \left( \frac{M}{2} (L-x) \right). \quad (12) \]

\[ |\sinh (\tau-x)\sqrt{\ln}| \leq \exp \left( \frac{M}{2} (\tau-x) \right). \quad (13) \]

Proof. For \( n \in \mathbb{Z} \setminus \{0\}, n < M \), one has

\[ |\cosh (L-x)\sqrt{\ln}| = \frac{\exp (L-x)\sqrt{\ln} + \exp -(L-x)\sqrt{\ln}}{2} \]
\[ \leq \frac{1}{2} \exp (L-x)\sqrt{\ln} + \frac{1}{2} \exp -(L-x)\sqrt{\ln} \leq \frac{1}{2} \exp \left( \frac{|n|}{2} (L-x) \right) + \frac{1}{2} \exp \left( -\frac{|n|}{2} (L-x) \right) \]
\[ \leq \frac{1}{2} \exp \left( \frac{M}{2} (L-x) \right) + \frac{1}{2} \leq \exp \left( \frac{M}{2} (L-x) \right). \]

and

\[ |\sinh (L-x)\sqrt{\ln}| = \frac{\exp (L-x)\sqrt{\ln} - \exp -(L-x)\sqrt{\ln}}{2\sqrt{\ln}} \]
\[ \leq \frac{\exp (L-x)\sqrt{\ln}}{2\sqrt{|n|}} + \frac{\exp -(L-x)\sqrt{\ln}}{2\sqrt{|n|}} \leq \frac{1}{2} \exp \left( \frac{|n|}{2} (L-x) \right) + \frac{1}{2} \exp \left( -\frac{|n|}{2} (L-x) \right) \]
\[ \leq \frac{1}{2} \exp \left( \frac{M}{2} (L-x) \right) + \frac{1}{2} \leq \exp \left( \frac{M}{2} (L-x) \right). \]

as required.

Lemma 2. For \( |n| > N_{x} \), we have

\[ Q_{n}(\phi^{\prime},\psi^{\prime},f^{\prime})(x) = \frac{1}{2} \left( u_{n}(x) - u_{n}^{’}(x) \right). \]

(14)

Proof. Differentiating (6) with respect to \( x \) gives

\[ -\frac{u_{n}^{’}(x)}{\sqrt{\ln}} = \sinh \left( (L-x)\sqrt{\ln} \right) \phi_{n} - \cosh \left( (L-x)\sqrt{\ln} \right) \psi_{n} - \frac{i}{\sqrt{\ln}} \int_{x}^{(\tau-x)\sqrt{\ln}} \frac{f_{n}^{’}(\tau) \, d\tau}{\sqrt{\ln}}. \]

(15)

Adding (15) to (6), we infer that
\[ u_N(x) - u'(x) = \exp((L-x)\sqrt{n})\varphi - \exp((L-x)\sqrt{n})\varphi' - \int_0^1 \exp(\tau-x)\sqrt{n} f_\tau(\tau) \, d\tau, \]
from which complete the proof.

The following theorem comes from the regularization \( u_N \), provides the error estimates in the \( L^2 \)-norm when the exact solution belongs to new spaces \( G^s \), \((s > 0)\). Here \( G^s \) is presented by
\[
G^s(0, 2\pi) = \left\{ \xi \in L^2(0, 2\pi) : \sum_{n \in \mathbb{Z}} |n|^{s} \exp \left| \int_0^{0.2\pi} \right| \left( |\xi(t)| \right) < \infty \right\},
\]
and this norm is given by
\[
\| \xi \|_{G^s(0, 2\pi)} = \left( \sum_{n \in \mathbb{Z}} |n|^{s} \exp \left| \int_0^{0.2\pi} \right| \left( |\xi(t)| \right) \right)^{1/2}.
\]

For a Hilbert space \( X \), we denote
\[
\| \xi \|_{L^\infty(0, L; X)} = \sup_{0 \leq t \leq L} \| \xi(t) \|_X,
\]
and
\[
\| \xi \|_{L^\infty(0, L; X)} = \sup_{0 \leq t \leq L} \| \xi(t) \|_X.
\]

Theorem 1. Assume that problem (1) has a weak solution \( u \in C([0, T]; L^2(0, 2\pi)) \). Choose \( N_\epsilon > 0 \) such that
\[
\lim_{\epsilon \to 0} N_\epsilon^{-1} \lim_{\epsilon \to 0} \exp \left( L \sqrt{\frac{N_\epsilon}{2}} \right) = 0.
\]
(a). Suppose that the problem (1) has a solution \( u \) satisfying
\[
\| u \|_{L^2([0, L; G^s(0, 2\pi)])} + \| u_n \|_{L^2([0, L; G^s(0, 2\pi)])} \leq E_1,
\]
for some known constant \( E_1 > 0 \). Then
\[
\| u_N(x, \cdot) - u(x, \cdot) \| \leq \sqrt{P^2 + 2\pi E_1} \exp \left( -x \sqrt{\frac{N_\epsilon}{2}} \right),
\]
where \( P = \sqrt{6\epsilon^2 \exp \left( L \sqrt{2N_\epsilon} \right) + \frac{6\epsilon^2 \exp \left( L \sqrt{2N_\epsilon} \right) - 1}{\sqrt{2N_\epsilon}}} \).
(b). Suppose that the problem (1) has a solution \( u \) satisfying
\[
\| u \|_{L^2([0, L; G^s(0, 2\pi)])} + \| u_n \|_{L^2([0, L; G^s(0, 2\pi)])} \leq E_2,
\]
for \( r \geq 0 \) and some known constant \( E_2 > 0 \). Then
\[
\| u_N(x, \cdot) - u(x, \cdot) \| \leq \sqrt{P^2 + 2\pi N_\epsilon^{-2r} E_2^2} \exp \left( -x \sqrt{\frac{N_\epsilon}{2}} \right).
\]

Corollary 1. Let us choose \( N_\epsilon = \frac{2}{(L+\delta)^2} \ln \left( \frac{1}{\epsilon} \right) \) for \( \delta > 0 \) then

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Estimate in (22) is calculated as follows

\[ \left\| u_N^e(x, \cdot) - u(x, \cdot) \right\| \leq \sqrt{R^2 + 2\pi E^2 \left( \frac{1}{\varepsilon} \right)^{1/2}}, \]

where \( R = \sqrt{\frac{2\delta}{6e^{L+\delta} + 6L \left( e^{L+\delta} - e^L \right)} \left[ \frac{2}{L+\delta} \ln \left( \frac{1}{\varepsilon} \right) \right]^{1/2}}. \)

2. Estimate in (24) is calculated as follows

\[ \left\| u_N^e(x, \cdot) - u(x, \cdot) \right\| \leq \sqrt{R^2 + 2\pi \left[ \frac{\sqrt{\varepsilon}}{L+\delta} \ln \left( \frac{1}{\varepsilon} \right) \right]^{1/2} E^2 \varepsilon^{1/2}.} \]

Proof of the Theorem 1. The proof is divided into two parts.

Part a. Estimate the error (22) between the regularization \( u_N^e \) and the exact solution \( u \) with a priori (21).

We rewrite \( u \) as

\[ u(x, t) = \sum_{k \in N, n=0}^{\infty} \left[ \cosh ((L-x)\sqrt{n}) \phi_n - \frac{\sinh ((L-x)\sqrt{n})}{\sqrt{n}} \psi_n - \frac{1}{\sqrt{n}} \frac{\sinh (\tau-x)\sqrt{n}}{\sqrt{n}} f_n(\tau) \right] \exp(\imath n t) + \Theta(\phi_n, \psi_n, f_n)(x) + \phi_N^* (\phi, \psi, f)(x, t). \]

From (11) and (27), thanks to Parseval’s relation, we obtain

\[ \left\| u_N^e(x, \cdot) - u(x, \cdot) \right\|^2 = 2\pi \sum_{k \in N, n=0}^{\infty} \left| u_N^e(x, u_n(x)) \right|^2 + 4\pi \sum_{k \in N, n=0}^{\infty} \left| \phi_N^*(\phi, \psi, f_n)(x) \right|^2 + 2\pi \left| \Theta(\phi_n, \psi_n, f_n)(x) \right|^2 + 4\pi \sum_{k \in N, n=0}^{\infty} \left| \phi_N^*(\phi, \psi, f_n)(x) \right|^2. \]

We now apply Lemma 1 and using the Holder’s inequality, we have

\[ J_1(x) \leq 6\pi \sum_{k \in N, n=0}^{\infty} \left[ \cosh ((L-x)\sqrt{n}) \right|^2 \left| \phi_n - \phi_n^e \right|^2 + \left| \sinh ((L-x)\sqrt{n}) \right|^2 \left| \psi_n - \psi_n^e \right|^2 \]

\[ + 6\pi \sum_{k \in N, n=0}^{\infty} \left| \frac{1}{\sqrt{n}} \frac{\sinh (\tau-x)\sqrt{n}}{\sqrt{n}} (f_n(\tau) - f_n(\tau)) \right|^2 \]

\[ \leq 6\pi \sum_{k \in N, n=0}^{\infty} \left[ \exp((L-x)\sqrt{2N_e}) \left| \phi_n - \phi_n^e \right|^2 + \exp((L-x)\sqrt{2N_e}) \left| \psi_n - \psi_n^e \right|^2 \right] \]

\[ + 6\pi \sum_{k \in N, n=0}^{\infty} \left[ (L-x) \exp((\tau-x)\sqrt{2N_e}) \left| f_n(\tau) - f_n(\tau) \right|^2 \right]. \]

where we have used the elementary inequality \( (a+b+c)^2 \leq 3(a^2 + b^2 + c^2). \)

Similarly, the second equation \( J_2(x) \) writes
\[ J_2(x) \leq 12\pi \sum_{k=1}^N \left[ \exp(-L-x) \right] \left[ \varphi_0 - \varphi_n \right]^2 + \left[ \exp(-L-x) \right] \left[ \varphi_0 - \psi_n \right]^2 + \left( \frac{1}{2} \right) \left[ \exp(-L-x) \right] \left[ \varphi_0 + \psi_n \right]^2 \]

\[ + 12\pi \sum_{k=1}^N \int \exp(-L-x) \varphi_n \psi_n \frac{d\tau}{\sqrt{\frac{\pi}{\sqrt{\frac{N^2}{2} + \frac{\pi}{\sqrt{\frac{N^2}{2}}}}}}} \]

\[ \leq 12\pi \sum_{k=1}^N \left[ \exp((-L-x)\sqrt{2N}) \right] \left[ \varphi_0 - \varphi_n \right]^2 + \left[ \exp((-L-x)\sqrt{2N}) \right] \left[ \varphi_0 - \psi_n \right]^2 \]

\[ + 12\pi \sum_{k=1}^N \left[ \exp((-L-x)\sqrt{2N}) \right] \left[ \varphi_n + \psi_n \right]^2 \].

Thanks to Holder’s inequality and using the basic inequality \( e^x \geq ax \), \( \forall a > 0 \), we deduce that

\[ J_2(x) = 6\pi \left[ \varphi_0 - \varphi_n \right]^2 + (L-x) \left[ \varphi_0 - \psi_n \right]^2 + (L-x) \left[ \varphi_n + \psi_n \right]^2 \]

\[ \leq 6\pi \left[ \exp((-L-x)\sqrt{2N}) \right] \left[ \varphi_0 - \varphi_n \right]^2 + \left[ \exp((-L-x)\sqrt{2N}) \right] \left[ \varphi_0 - \psi_n \right]^2 \]

\[ + 6\pi \left[ \exp((-L-x)\sqrt{2N}) \right] \left[ \varphi_n + \psi_n \right]^2 \].

Using Lemma 2, easy calculations show that

\[ J_2(x) = 4\pi \sum_{k=1}^N \left[ \varphi_0 \psi_n (x, \varphi_0, f)(x) \right]^2 = 4\pi \sum_{k=1}^N \left[ \frac{1}{2} \left[ \varphi_0 \psi_n (x) - \frac{u_0 (x)}{\sqrt{\frac{\pi}{\sqrt{\frac{N^2}{2} + \frac{\pi}{\sqrt{\frac{N^2}{2}}}}}}} \right]^2 \right] \]

\[ \leq \pi \sum_{k=1}^N \left[ \exp(-\sqrt{2N}) \left[ \exp(-\sqrt{\frac{\pi}{\sqrt{\frac{N^2}{2} + \frac{\pi}{\sqrt{\frac{N^2}{2}}}}}}) \right] \left[ \varphi_0 \psi_n (x) - \frac{u_0 (x)}{\sqrt{\frac{\pi}{\sqrt{\frac{N^2}{2} + \frac{\pi}{\sqrt{\frac{N^2}{2}}}}}}} \right]^2 \right] \]

\[ \leq \pi \sum_{k=1}^N \left[ \exp(-\sqrt{2N}) \left[ \exp(-\sqrt{\frac{\pi}{\sqrt{\frac{N^2}{2} + \frac{\pi}{\sqrt{\frac{N^2}{2}}}}}}) \right] \left[ \varphi_0 \psi_n (x) - \frac{u_0 (x)}{\sqrt{\frac{\pi}{\sqrt{\frac{N^2}{2} + \frac{\pi}{\sqrt{\frac{N^2}{2}}}}}}} \right]^2 \right] \]

\[ \leq 2\pi \exp(-\sqrt{2N}) \left[ \sum_{k=1}^N \left[ \exp(-\frac{\pi}{\sqrt{\frac{N^2}{2} + \frac{\pi}{\sqrt{\frac{N^2}{2}}}}}) \right] \left[ \varphi_0 \psi_n (x) - \frac{u_0 (x)}{\sqrt{\frac{\pi}{\sqrt{\frac{N^2}{2} + \frac{\pi}{\sqrt{\frac{N^2}{2}}}}}}} \right]^2 \right] \]

\[ \leq 2\pi \exp(-\sqrt{2N}) \left[ \|u_0\|_{L^1(\mathbb{R}, L^2(0,2\pi))}^2 \right].

Combining (28), (29), (30), (31) and (32) we infer

\[ \left[ \varphi_0 (x, \cdot) - u_0 (x) \right]^2 \leq 6\exp((-L-x)\sqrt{2N}) \left[ \varphi_0 - \varphi_n \right]^2 + \left[ \varphi_0 - \psi_n \right]^2 + \left[ \exp((-L-x)\sqrt{2N}) \right] \left[ f_0 (x, \cdot) - f_0 (x, \cdot) \right]^2 \]

\[ + 2\pi \exp(-\sqrt{2N}) \left[ \left[ \exp(-\frac{\pi}{\sqrt{\frac{N^2}{2} + \frac{\pi}{\sqrt{\frac{N^2}{2}}}}}) \right] \left[ \varphi_0 \psi_n (x) - \frac{u_0 (x)}{\sqrt{\frac{\pi}{\sqrt{\frac{N^2}{2} + \frac{\pi}{\sqrt{\frac{N^2}{2}}}}}}} \right]^2 \right] \]

\[ \leq 6\exp((-L-x)\sqrt{2N}) \left[ \varphi_0 - \varphi_n \right]^2 + \left[ \varphi_0 - \psi_n \right]^2 + \left[ \exp((-L-x)\sqrt{2N}) \right] \left[ f_0 (x, \cdot) - f_0 (x, \cdot) \right]^2 \]

\[ + 2\pi \exp(-\sqrt{2N}) \left[ \left[ \exp(-\frac{\pi}{\sqrt{\frac{N^2}{2} + \frac{\pi}{\sqrt{\frac{N^2}{2}}}}}) \right] \left[ \varphi_0 \psi_n (x) - \frac{u_0 (x)}{\sqrt{\frac{\pi}{\sqrt{\frac{N^2}{2} + \frac{\pi}{\sqrt{\frac{N^2}{2}}}}}}} \right]^2 \right] \]

\[ \leq 6\exp((-L-x)\sqrt{2N}) \left[ \varphi_0 - \varphi_n \right]^2 + \left[ \varphi_0 - \psi_n \right]^2 + \left[ \exp((-L-x)\sqrt{2N}) \right] \left[ f_0 (x, \cdot) - f_0 (x, \cdot) \right]^2 + 2\pi \exp(-\sqrt{2N}) \left[ 1 - \exp((-L-x)\sqrt{2N}) \right] \]

\[ \leq 6\exp((-L-x)\sqrt{2N}) \left[ \varphi_0 - \varphi_n \right]^2 + \left[ \varphi_0 - \psi_n \right]^2 + \left[ \exp((-L-x)\sqrt{2N}) \right] \left[ f_0 (x, \cdot) - f_0 (x, \cdot) \right]^2 + 2\pi \exp(-\sqrt{2N}) \left[ 1 - \exp((-L-x)\sqrt{2N}) \right] \]

\[ \leq 6\exp((-L-x)\sqrt{2N}) \left[ \varphi_0 - \varphi_n \right]^2 + \left[ \varphi_0 - \psi_n \right]^2 + \left[ \exp((-L-x)\sqrt{2N}) \right] \left[ f_0 (x, \cdot) - f_0 (x, \cdot) \right]^2 + 2\pi \exp(-\sqrt{2N}) \left[ 1 - \exp((-L-x)\sqrt{2N}) \right] \]

which can be rewritten as

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\[
\left\| u_\varepsilon(x, \cdot) - u(x, \cdot) \right\| \leq \left[ 6\varepsilon^2 \exp(L\sqrt{2N_x}) + \frac{6L\varepsilon^2 \left( \exp(L\sqrt{2N_x}) - 1 \right) + 2\pi E_1^2}{\sqrt{2N_x}} \right]^{\frac{1}{2}} \exp\left(-x\sqrt{\frac{N_x}{2}}\right).
\]

(34)

Part (b). Estimate the error (24) between the regularization \( u_\varepsilon \) and the exact solution \( u \) with an error (23).

By an argument analogous to the previous one, the estimates of \( J_1(x), J_2(x), J_3(x) \) in the proof of part (a) remains valid. Also, replace \( J_4(x) \) by the following estimate:

\[
J_4(x) = 4\pi \sum_{k \in \mathbb{N}^m} |Q_{x, \varepsilon}^p(\varphi, \psi, f)(x)|^2 = 4\pi \sum_{k \in \mathbb{N}^m} \left\| u_\varepsilon(x) \right\|^2 - u_\varepsilon(x, \cdot) \right\|^2 \right\| 
\leq \frac{\pi}{4L^2} \sum_{k \in \mathbb{N}^m} \left\| \nabla u_\varepsilon(x) \right\|^2 \| \nabla^2 u_\varepsilon(x) \| \exp\left(-x\sqrt{\frac{N_x}{2}}\right) \left\| u_\varepsilon(x) \right\|^2 \left\| u_\varepsilon(x, \cdot) \right\|^2 
\leq 2\pi N_x^{-2\varepsilon} \exp\left(-x\sqrt{\frac{N_x}{2}}\right) \left\{ \sum_{k \in \mathbb{N}^m} \left\| \nabla u_\varepsilon(x) \right\|^2 \left\| u_\varepsilon(x, \cdot) \right\|^2 \right\} 
\leq 2\pi N_x^{-2\varepsilon} \exp\left(-x\sqrt{\frac{N_x}{2}}\right) \left\{ \left\| u_\varepsilon \right\|^2_{L^2((0, L), L^2(0, 2\pi))} + \left\| u_\varepsilon \right\|^2_{L^2((0, L), L^2(0, 2\pi))} \right\}.
\]

Combining (28), (29), (30), (31) and (35), we get

\[
\left\| u_\varepsilon(x, \cdot) - u(x, \cdot) \right\| \leq \left[ 6\varepsilon^2 \exp(L\sqrt{2N_x}) + \frac{6L\varepsilon^2 \left( \exp(L\sqrt{2N_x}) - 1 \right) + 2\pi E_1^2}{\sqrt{2N_x}} \right]^{\frac{1}{2}} \exp\left(-x\sqrt{\frac{N_x}{2}}\right).
\]

(36)

This completes the proof of the theorem.

**CONCLUSION**

In this paper, the Cauchy problem for the heat equation has been solved by employing the truncation method for a resulting linear integral equation. Convergence and stability estimates, as the regularization parameter tends to zero, are proved.
Chính hóa bài toán Cauchy cho phương trình nhiệt

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TÓM TÁT
Trong bài báo này, chúng tôi nghiên cứu bài toán Cauchy cho phương trình nhiệt với hàm nguồn tuyến tính thỏa phương trình:
\[ u_t (x,t) = u_{xx}(x,t) + f(x,t), \quad u(x,0) = \phi(x), \quad u_x (x,0) = \psi(x), \quad (x,t) \in (0, L) \times (0, T). \]
Đây là bài toán không chỉnh theo nghĩa của Hadamard. Để chính hóa bài toán này, phương pháp chặt cụt được đề xuất để giải quyết bài toán trong trường hợp dữ liệu Cauchy \( \phi, \psi \) và hàm nguồn \( f \) bị nhiễu bởi \( \xi, \eta \) và \( f \xi \) thỏa mãn
\[ \| \phi - \phi^* \| + \| \psi - \psi^* \| \leq \varepsilon \quad \text{và} \quad \| f\xi(x,\cdot) - f(x,\cdot) \| \leq \varepsilon. \]
Chúng tôi đưa ra các đánh giá sai số giữa nghiệm chính hóa và nghiệm chính xác dưới một số tính trơn khác nhau của nghiệm chính xác.

Từ khóa: phương trình Elliptic, bài toán không chỉnh, bài toán Cauchy, phương pháp chính hóa, phương pháp chặt cụt

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