Regularization for a Riesz-Feller space fractional backward diffusion problem with a time-dependent coefficient

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(Received on 5th December 2016, accepted on 28 th November 2017)

ABSTRACT

In the present paper, we consider a backward problem for a space-fractional diffusion equation (SFDE) with a time-dependent coefficient. Such the problem is obtained from the classical diffusion equation by replacing the second-order spatial derivative with the Riesz-Feller derivative of order $\alpha \in (0,2]$. *This problem is ill-posed, i.e., the solution (if it exists) does not depend continuously on the data. Therefore, we propose one new regularization solution to solve it. Then, the convergence estimate is obtained under a priori bound assumptions for exact solution.*

***Key words**: space-fractional backward diffusion problem, Ill-posed problem, Regularization, error estimate, time-dependent coefficient*

INTRODUCTION

The fractional differential equations appear more and more frequently in physical, chemical, biology and engineering applications. Nowadays, fractional diffusion equation plays important roles in modeling anomalous diffusion and subdiffusion systems [2], description of fractional random walk, unification of diffusion [3], and wave propagation phenomenon [4]. It is well known that the SFDE is obtained from the classical diffusion equation in which the second-order space derivative is replaced with a space-fractional partial derivative.

Let $\vartheta : [0,T] \to \mathbb{R}$ is a continuous function on $[0,T]$ satisfying $\vartheta(t) > 0$. In this paper, we consider a backward problem for the following nonlinear SFDE with a time-dependent coefficient

$$\begin{cases} u_t(x,t) = \vartheta(t)\, {}_xD_\theta^\alpha + F(x,t,u(x,t)), (x,t) \in \mathbb{R} \times (0,T), \\ u(x,t)\big|_{x \to \pm\infty} = 0, t \in (0,T), \\ u(x,\mathrm{T}) = G(x), x \in \mathbb{R}, \end{cases} \tag{1}$$

where the fractional spatial derivative ${}_xD_\theta^\alpha$ is the Riesz-Feller fractional derivative of order $\alpha (0 < \alpha \le 2)$ and skewness $\theta\ (|\theta| \le \min\{\alpha, 2-\alpha\}, \theta \ne \pm 1)$ defined in [5], as follows:

$$\begin{cases} {}_xD_\theta^\alpha f(x) = \dfrac{\Gamma(1+\alpha)}{\pi}\left\{\sin\dfrac{(\alpha+\theta)\pi}{2}\displaystyle\int_0^\infty \dfrac{f(x+s)-f(x)}{s^{1+\alpha}}ds + \sin\dfrac{(\alpha-\theta)\pi}{2}\int_0^\infty \dfrac{f(x-s)-f(x)}{s^{1+\alpha}}ds\right\}, 0<\alpha<2, \\ {}_xD_0^\alpha f(x) = \dfrac{d^2 f(x)}{dx^2}, \alpha = 2. \end{cases}$$

Here, we wish to determine the temperature $u(x,t)$ from temperature measurements $G^\varepsilon(x)$. Since the measurements usually contain an error, we now could assume that the measured data function $G^\varepsilon(x)$ satisfies $\left\|G - G^\varepsilon\right\|_{L^2(\mathbb{R})} \le \varepsilon$, where the constant $\varepsilon > 0$ represents the noise level. Moreover, assume there hold the following a priori bound

$$\left\|u(\cdot,0)\right\|_{L^2(\mathbb{R})} \le E, E > \varepsilon. \tag{2}$$

We assume that F satisfies the Lipschitz condition

$$\left\|F(x,t,z_1) - F(x,t,z_2)\right\|_{L^2(\mathbb{R})} \le K_F \left\|z_1 - z_2\right\|_{L^2(\mathbb{R})} \tag{3}$$

for some constant K_F independent of x, t, z_1, z_2 with

$$K_F \in \left[0, \frac{1}{T}\right). \tag{4}$$

In case of the source function $F = 0$ and $\vartheta(t) = 1$, Problem (1) has been proposed by some authors. Zheng and Wei [7] used two methods, the spectral regularization and modified equation methods, to solve this problem. In [6], they developed an optimal modified method to solve this problem by an a priori and an a posteriori strategy. In 2014, Zhao et al [8] applied a simplified Tikhonov regularization method to deal with this problem. After then, a new regularization method of iteration type for solving this problem has been introduced by Cheng et al [1]. Although we have many works on the linear homogeneous case of the backward problem, the nonlinear case of the problem is quite scarce. For the nonlinear problem, the solution u is complicated and defined by an integral equation such that the right hand side depends on u. This leads to studying nonlinear problem is very difficult, so in this paper we develop a new appropriate technique.

The remainder of this paper is organized as follows. In Section 2, we propose the regularizing scheme for Problem (1). Then, in Section 3, we show that the regularizing scheme of Problem (1) is well-posed. Finally, the convergence estimate is given in Section 4.

REGULARIZATION FOR PROBLEM (1)

Let $\hat{G}(\omega)$ denote the Fourier transform of the integrable function $G(x)$, which defined by

$$\hat{G}(\omega) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \exp(-ix\omega)G(x)dx, \quad i = \sqrt{-1}.$$

In terms of the Fourier transform, we have the following properties for the Riesz-Feller space-fractional derivative [5]

$${}_xD_\theta^\alpha(G)(\omega) = -\psi_\alpha^\theta(\omega)\hat{G}(\omega),$$

where

$$\psi_\alpha^\theta(\omega) = |\omega|^\alpha \left[\cos\left(\frac{\theta\pi}{2}\right) + i\,\mathrm{sign}(\omega)\sin\left(\frac{\pi\theta}{2}\right)\right]. \tag{5}$$

We define the function $k(t)$ by $k(t)=\int_0^t \frac{1}{\vartheta(s)}ds.$

By taking a Fourier transform to Problem (1), we transform Problem (1) into the following differential equation

$$\begin{cases} \hat{u}_t(\omega,t) = -\vartheta(t)\psi_\alpha^\theta(\omega)\hat{u}(\omega,t)+\hat{F}(\omega,t,u(\omega,\mathrm{t})), \\ \hat{u}(\omega,\mathrm{T}) = \hat{G}(\omega). \end{cases} \tag{6}$$

The solution to equation (6) is given by

$$\hat{u}(\omega,t)=\exp\big(\psi_\alpha^\theta(\omega)(k(T)-k(t))\big)\Big[\hat{G}(\omega)-\int_t^T \exp\big(\psi_\alpha^\theta(\omega)(k(s)-k(T))\big)\hat{F}(\omega,s,u(\omega,s))ds\Big]. \tag{7}$$

From (7), applying the inverse Fourier transform, we get

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\big(\psi_\alpha^\theta(\omega)(k(T)-k(t))\big)\Big[\hat{G}(\omega)-\int_t^T \exp\big(\psi_\alpha^\theta(\omega)(k(s)-k(T))\big)\hat{F}(\omega,s,u(\omega,s))ds\Big]\exp(ix\omega)d\omega. \tag{8}$$

From which when $|\omega|$ becomes large, the terms $\left|\exp\left(\psi_\alpha^\theta(\omega)(k(T)-k(t))\right)\right|$ increases rather quickly: small errors in high-frequency components can blow up and completely destroy the solution for $0<t<T$, therefore recovering the scalar (temperature, pollution) $u(x,t)$ from the measured data $G^\varepsilon(x)$ is severely ill-posed. In this note, we regularize Problem (1) by the problem

$$\begin{aligned} U_\beta^\varepsilon(\omega,t) &= \frac{\exp\left(\psi_\alpha^\theta(\omega)(k(T)-k(t))\right)}{1+\beta\exp\left(|\omega|^\alpha\cos\left(\frac{\theta\pi}{2}\right)k(T)\right)}G^\varepsilon(\omega)-\int_t^T\frac{\exp\left(\psi_\alpha^\theta(\omega)(k(s)-k(t))\right)}{1+\beta\exp\left(|\omega|^\alpha\cos\left(\frac{\theta\pi}{2}\right)k(T)\right)}\hat{F}(\omega,s,U_\beta^\varepsilon(\omega,s))ds \\ &+\int_0^t\frac{\beta\exp\left(|\omega|^\alpha\cos\left(\frac{\theta\pi}{2}\right)k(T)\right)}{1+\beta\exp\left(|\omega|^\alpha\cos\left(\frac{\theta\pi}{2}\right)k(T)\right)}\exp\left((k(s)-k(t))\psi_\alpha^\theta(\omega)\right)\hat{F}(\omega,s,U_\beta^\varepsilon(\omega,s))ds, \end{aligned} \tag{9}$$

where β is regularization parameter.

THE WELL POSEDNESS OF PROBLEM (9)

First, we consider the following Lemma which is used in the proof of the main results.

Lemma 1. *Let* $t,s\in[0,T]$.

1) If $s\ge t,$ *then we have*

$$a)\left|\frac{\exp\left(\psi_\alpha^\theta(\omega)(k(s)-k(t))\right)}{1+\beta\exp\left(|\omega|^\alpha\cos\left(\frac{\pi\theta}{2}\right)k(T)\right)}\right|\le\beta^{\frac{k(t)-k(s)}{k(T)}}.$$

$$b)\left|\frac{\exp\left(\psi_\alpha^\theta(\omega)(k(T)-k(t))\right)}{1+\beta\exp\left(|\omega|^\alpha\cos\left(\frac{\pi\theta}{2}\right)k(T)\right)}\right|\le\beta^{\frac{k(t)-k(T)}{k(T)}}.$$

2) If $s\le t$, *then we have*

$$c)\left|\frac{\beta\exp\left(\psi_\alpha^\theta(\omega)(k(s)-k(t)+k(T))\right)}{1+\beta\exp\left(|\omega|^\alpha\cos\left(\frac{\theta\pi}{2}\right)k(T)\right)}\right|\le\beta^{\frac{k(t)-k(s)}{k(T)}}.$$

Proof. First, we prove (a). In fact, we have

$$\left|\frac{\exp\left(\psi_\alpha^\theta(\omega)(k(s)-k(t))\right)}{1+\beta\exp\left(|\omega|^\alpha\cos(\frac{\pi\theta}{2})k(T)\right)}\right|=\frac{\exp\left(|\omega|^\alpha\cos(\frac{\pi\theta}{2})(k(s)-k(t)-k(T))\right)}{\beta+\exp\left(-|\omega|^\alpha\cos(\frac{\pi\theta}{2})k(T)\right)}$$

$$=\frac{\exp\left(|\omega|^\alpha\cos(\frac{\pi\theta}{2})(k(s)-k(t)-k(T))\right)}{\left[\beta+\exp\left(-|\omega|^\alpha\cos(\frac{\pi\theta}{2})k(T)\right)\right]^{\frac{k(s)-k(t)}{k(T)}}\left[\beta+\exp\left(-|\omega|^\alpha\cos(\frac{\pi\theta}{2})k(T)\right)\right]^{\frac{k(T)-k(s)+k(t)}{k(T)}}}$$

$$\le\frac{1}{\left[\beta+\exp\left(-|\omega|^\alpha\cos(\frac{\pi\theta}{2})k(T)\right)\right]^{\frac{k(s)-k(t)}{k(T)}}}\le\beta^{\frac{k(t)-k(s)}{k(T)}}.$$

As an immediate consequence of (a), making the change $s=T$, we have (b).
Next, we prove (c). In fact from (b), we obtain

$$\left|\frac{\exp(\psi_\alpha^\theta(\omega)(k(T)-(k(t)-k(s))))}{1+\beta\exp(|\omega|^\alpha\cos(\frac{\pi\theta}{2})k(T))}\right|\le\beta^{\frac{k(t)-k(s)-k(T)}{k(T)}}$$

it follows that

$$\left|\frac{\beta\exp\left(\psi_\alpha^\theta(\omega)(k(s)-k(t)+k(T))\right)}{1+\beta\exp\left(|\omega|^\alpha\cos(\frac{\pi\theta}{2})k(T)\right)}\right|\le\beta^{\frac{k(t)-k(s)}{k(T)}}.$$

This completes the proof. □

We are now in a position to prove the following theorem.

Theorem 1. *Suppose* $m \in \left(0, \frac{1}{K_F^2 T^2} - 1\right)$. *Let* $G \in L^2(\mathbb{R})$ *and* F *satisfies (3) then Problem (9) is well-posed.*

Proof. We divide it into two steps.

Step1. The existence and the uniqueness of a solution of Problem (9).

Let us define the norm on $C([0;T];L^2(\mathbb{R}))$ as follows

$$\| h\|_0 = \sup_{0\le t\le T} \beta^{\frac{-k(t)}{k(T)}} \| h(t)\|_{L^2(\mathbb{R})}, \text{ for all } h \in C([0;T];L^2(\mathbb{R})).$$

It is easily be seen that $\| \cdot\|_0$ is a norm of $C([0;T];L^2(\mathbb{R}))$.

For $v \in C([0;T];L^2(\mathbb{R}))$, we consider the following function

$$A(v)(x,t) = \frac{1}{\sqrt{2\pi}} B(x,t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty}\int_t^T \frac{\exp\left(\psi_\alpha^\theta(\omega)(k(s)-k(t))\right)}{1+\beta\exp\left(|\omega|^\alpha \cos\left(\frac{\pi\theta}{2}\right)k(T)\right)} \hat{F}(\omega,s,v)\exp(i\omega x)\,ds\,d\omega$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty}\int_0^t \frac{\beta\exp\left(|\omega|^\alpha \cos\left(\frac{\pi\theta}{2}\right)k(T)\right)}{1+\beta\exp\left(|\omega|^\alpha \cos\left(\frac{\pi\theta}{2}\right)k(T)\right)} \exp\left((k(s)-k(t))\psi_\alpha^\theta(\omega)\right)\hat{F}(\omega,s,v)\exp(i\omega x)\,ds\,d\omega,$$

where

$$B(x,t) = \int_{-\infty}^{+\infty} \frac{\exp\left(\psi_\alpha^\theta(\omega)(k(T)-k(t))\right)}{1+\beta\exp\left(|\omega|^\alpha \cos\left(\frac{\pi\theta}{2}\right)k(T)\right)} G^\varepsilon(\omega)\exp(i\omega x)\,d\omega.$$

We claim that, for every $v_1, v_2 \in C([0;T];L^2(\mathbb{R}))$

$$\| A(v_1) - A(v_2)\|_0 \le K_F T \| v_1 - v_2\|_0 . \tag{10}$$

First, by Lemma 1 and (3), we have two following estimates for all $t \in [0,T]$

$$J_1 = \int_{-\infty}^{+\infty} \left(\int_t^T \frac{\exp\left(\psi_\alpha^\theta(\omega)(k(s)-k(t))\right)}{1+\beta\exp\left(|\omega|^\alpha \cos\left(\frac{\pi\theta}{2}\right)k(T)\right)} \left(\hat{F}(\omega,s,v_1) - \hat{F}(\omega,s,v_2)\right) ds \right)^2 d\omega$$

$$\leq (T-t)\int_{-\infty}^{+\infty}\int_t^T \left|\frac{\exp\left(\psi_\alpha^\theta(\omega)(k(s)-k(t))\right)}{1+\beta\exp\left(|\omega|^\alpha \cos\left(\frac{\pi\theta}{2}\right)k(T)\right)}\right|^2 \left|\hat{F}(\omega,s,v_1)-\hat{F}(\omega,s,v_2)\right|^2 dsd\omega$$

$$\leq (T-t)\int_{-\infty}^{+\infty}\int_t^T \beta^{\frac{2(k(t)-k(s))}{k(T)}} \left|\hat{F}(\omega,s,v_1)-\hat{F}(\omega,s,v_2)\right|^2 dsd\omega \leq \beta^{\frac{2k(t)}{k(T)}} K_F^2 (T-t)\int_t^T \beta^{\frac{-2k(s)}{k(T)}} \| v_1(\cdot,s)-v_2(\cdot,s)\|_{L^2(\mathbb{R})}^2 ds$$

$$\leq \beta^{\frac{2k(t)}{k(T)}} K_F^2 (T-t)^2 \sup_{0\leq s\leq T} \beta^{\frac{-2k(s)}{k(T)}} \| v_1(\cdot,s)-v_2(\cdot,s)\|_{L^2(\mathbb{R})}^2 \leq \beta^{\frac{2k(t)}{k(T)}} K_F^2 (T-t)^2 \| v_1-v_2\|_0^2 \tag{11}$$

and

$$J_2 = \int_{-\infty}^{+\infty}\left(\int_0^t \frac{\beta\exp\left(|\omega|^\alpha \cos\left(\frac{\pi\theta}{2}\right)k(T)\right)}{1+\beta\exp\left(|\omega|^\alpha \cos\left(\frac{\pi\theta}{2}\right)k(T)\right)} \exp\left((k(s)-k(t))\psi_\alpha^\theta(\omega)\right) \left(\hat{F}(\omega,s,v_1)-\hat{F}(\omega,s,v_2)\right) ds\right)^2 d\omega$$

$$\leq t\int_{-\infty}^{+\infty}\int_0^t \left|\frac{\beta\exp\left(|\omega|^\alpha \cos\left(\frac{\pi\theta}{2}\right)k(T)\right)}{1+\beta\exp\left(|\omega|^\alpha \cos\left(\frac{\pi\theta}{2}\right)k(T)\right)} \exp\left((k(s)-k(t))\psi_\alpha^\theta(\omega)\right)\right|^2 \left|\hat{F}(\omega,s,v_1)-\hat{F}(\omega,s,v_2)\right|^2 dsd\omega$$

$$\leq t\int_{-\infty}^{+\infty}\int_0^t \beta^{\frac{2(k(t)-k(s))}{k(T)}} \left|\hat{F}(\omega,s,v_1)-\hat{F}(\omega,s,v_2)\right|^2 dsd\omega \leq \beta^{\frac{2k(t)}{k(T)}} K_F^2 t\int_0^t \beta^{\frac{-2k(s)}{k(T)}} \| v_1(\cdot,s)-v_2(\cdot,s)\|_{L^2(\mathbb{R})}^2 ds \tag{12}$$

$$\leq \beta^{\frac{2k(t)}{k(T)}} K_F^2 t^2 \sup_{0\leq s\leq T} \beta^{\frac{-2k(s)}{k(T)}} \| v_1(.,s)-v_2(.,s)\|_{L^2(\mathbb{R})}^2 \leq \beta^{\frac{2k(t)}{k(T)}} K_F^2 t^2 \| v_1-v_2\|_0^2 .$$

For $0<t<T$, using the inequality $(a+b)^2 \leq (1+m)a^2 + \left(1+\frac{1}{m}\right)b^2$ for all real numbers a and b and $m>0$, we obtain

$$\left\|A(v_1)(\cdot,t)-A(v_2)(\cdot,t)\right\|_{L^2(\mathbb{R})}^2 \leq (1+m)\beta^{\frac{2k(t)}{k(T)}} K_F^2 t^2 \| v_1-v_2\|_0^2 + \left(1+\frac{1}{m}\right)\beta^{\frac{2k(t)}{k(T)}} K_F^2 (T-t)^2 \| v_1-v_2\|_0^2 .$$

By choosing $m=\frac{T-t}{t}$, we have

$$\beta^{\frac{-2k(t)}{k(T)}} \| A(v_1)(\cdot,t)-A(v_2)(\cdot,t)\|_{L^2(\mathbb{R})}^2 \leq K_F^2 T^2 \| v_1-v_2\|_0^2, \text{ for all } t\in(0,T). \tag{13}$$

On the other hand, letting $t=0$ in (11), we have

$$\| A(v_1)(\cdot,0)-A(v_2)(\cdot,0)\|_{L^2(\mathbb{R})}^2 \leq K_F^2 T^2 \| v_1-v_2\|_0^2 . \tag{14}$$

By letting $t=T$ in (12), we have

$$\beta^{-2}\| A(v_1)(\cdot,T)-A(v_2)(\cdot,T)\|_{L^2(\mathbb{R})}^2 \leq K_F^2 T^2 \| v_1-v_2\|_0^2 . \tag{15}$$

Combining (13), (14) and (15), we obtain

$$\beta^{\frac{-2k(t)}{k(T)}} \| A(v_1)(\cdot,t) - A(v_2)(\cdot,t)\|^2_{L^2(\mathbb{R})} \le K_F^2 T^2 \| v_1 - v_2\|_0^2, \text{ for all } t \in [0,T]$$

which leads to (10). Since $K_F T < 1, A$ is a contraction. It follows that the equation $A(v) = v$ has a unique solution $U_\beta^\varepsilon \in C([0;T]; L^2(\mathbb{R}))$.

Step 2. The solution of Problem (9) continuously depends on the data.

Let $V_\beta^\varepsilon, W_\beta^\varepsilon$ be two solutions of Problem (9) corresponding to the final values G_V and G_W. By straightforward computation, we write

$$\left|W_\beta^\varepsilon(\omega,t) - V_\beta^\varepsilon(\omega,t)\right| \le \left| \frac{\exp\left(\psi_\alpha^\theta(\omega)(k(T)-k(t))\right)}{1+\beta\exp\left(|\omega|^\alpha \cos\left(\frac{\pi\theta}{2}\right)k(T)\right)} \left(G_W(\omega) - G_V(\omega)\right) \right|$$

$$+\left| \int_t^T \frac{\exp\left(\psi_\alpha^\theta(\omega)(k(s)-k(t))\right)}{1+\beta\exp\left(|\omega|^\alpha \cos\left(\frac{\pi\theta}{2}\right)k(T)\right)} \left[\hat{F}(\omega,s,V_\beta^\varepsilon(\omega,s)) - \hat{F}(\omega,s,W_\beta^\varepsilon(\omega,s))\right] ds \right|$$

$$+\left| \int_0^t \frac{\beta\exp(|\omega|^\alpha \cos(\frac{\pi\theta}{2})k(T))}{1+\beta\exp(|\omega|^\alpha \cos(\frac{\pi\theta}{2})k(T))} \exp((k(s)-k(t))\psi_\alpha^\theta(\omega))\, [\hat{F}(\omega,s,W_\beta^\varepsilon(\omega,s)) - \hat{F}(\omega,s,V_\beta^\varepsilon(\omega,s))] ds \right|.$$

Now applying Lemma 1, we get

$$\left|W_\beta^\varepsilon(\omega,t) - V_\beta^\varepsilon(\omega,t)\right| \le \beta^{\frac{k(t)-k(T)}{k(T)}} \left|G_W(\omega) - G_V(\omega)\right| + \int_t^T \beta^{\frac{k(t)-k(s)}{k(T)}} \left|\hat{F}(\omega,s,V_\beta^\varepsilon(\omega,s)) - \hat{F}(\omega,s,W_\beta^\varepsilon(\omega,s))\right| ds$$

$$+\int_0^t \beta^{\frac{k(t)-k(s)}{k(T)}} \left|\hat{F}(\omega,s,V_\beta^\varepsilon(\omega,s)) - \hat{F}(\omega,s,W_\beta^\varepsilon(\omega,s))\right| ds$$

$$\le \beta^{\frac{k(t)-k(T)}{k(T)}} \left|G_W(\omega) - G_V(\omega)\right| + \int_0^T \beta^{\frac{k(t)-k(s)}{k(T)}} \left|\hat{F}(\omega,s,V_\beta^\varepsilon(\omega,s)) - \hat{F}(\omega,s,W_\beta^\varepsilon(\omega,s))\right| ds.$$

Since $m \in \left(0, \frac{1}{K_F^2 T^2} - 1\right)$, we have that $0 < K_F < \frac{1}{T\sqrt{1+m}}$. From the inequality

$$(a+b)^2 \le \left(1+\frac{1}{m}\right)a^2 + (1+m)b^2 \tag{16}$$

for all real number a,b and $m > 0$, we get

$$\left\|W_\beta^\varepsilon(\cdot,t) - V_\beta^\varepsilon(\cdot,t)\right\|^2_{L^2(\mathbb{R})} = \left\|W_\beta^\varepsilon(\cdot,t) - V_\beta^\varepsilon(\cdot,t)\right\|^2_{L^2(\mathbb{R})}$$

$$\le \left(1+\frac{1}{m}\right)\beta^{\frac{2k(t)-2k(T)}{k(T)}}\left\|G_W-G_V\right\|^2_{L^2(\mathbb{R})}+(1+m)\int_{-\infty}^{+\infty}\left(\int_0^T\beta^{\frac{k(t)-k(s)}{k(T)}}\left|\hat{F}(\omega,s,V^{\varepsilon}_{\beta}(\omega,s))-\hat{F}(\omega,s,W^{\varepsilon}_{\beta}(\omega,s))\right|ds\right)^2 d\omega$$

$$\le \left(1+\frac{1}{m}\right)\beta^{\frac{2k(t)-2k(T)}{k(T)}}\left\|G_W-G_V\right\|^2_{L^2(\mathbb{R})}+(1+m)K_F^2T\int_0^T\beta^{\frac{2k(t)-2k(s)}{k(T)}}\left\|W^{\varepsilon}_{\beta}(\cdot,s)-V^{\varepsilon}_{\beta}(\cdot,s)\right\|^2_{L^2(\mathbb{R})}ds.$$

This leads to

$$\beta^{\frac{-2k(t)}{k(T)}}\left\|W^{\varepsilon}_{\beta}(\cdot,t)-V^{\varepsilon}_{\beta}(\cdot,t)\right\|^2_{L^2(\mathbb{R})}\le\left(1+\frac{1}{m}\right)\beta^{-2}\left\|G_W-G_V\right\|^2_{L^2(\mathbb{R})}+(1+m)K_F^2T\int_0^T\beta^{\frac{-2k(s)}{k(T)}}\left\|W^{\varepsilon}_{\beta}(\cdot,s)-V^{\varepsilon}_{\beta}(\cdot,s)\right\|^2_{L^2(\mathbb{R})}ds. \quad (17)$$

Set $Z(t)=\beta^{\frac{-2k(t)}{k(T)}}\| W^{\varepsilon}_{\beta}(\cdot,t)-V^{\varepsilon}_{\beta}(\cdot,t)\|^2_{L^2(\mathbb{R})}, \forall t\in[0,T]$. Since $W^{\varepsilon}_{\beta},V^{\varepsilon}_{\beta}\in C([0,T];L^2(\mathbb{R}))$, we see that the function Z is continuous on $[0,T]$ and attains over there its maximum M at some $t_0\in[0,T]$. Let $M=\max_{t\in[0,T]}Z(t)$. From (17), we obtain

$$M\le\left(1+\frac{1}{m}\right)\beta^{-2}\left\|G_W-G_V\right\|^2_{L^2(\mathbb{R})}+(1+m)K_F^2T^2M,$$

or equivalently

$$\left[1-(1+m)K_F^2T^2\right]M\le\left(1+\frac{1}{m}\right)\beta^{-2}\left\|G_W-G_V\right\|^2_{L^2(\mathbb{R})}.$$

This implies that for all $t\in[0,T]$

$$\beta^{\frac{-2k(t)}{k(T)}}\left\|W^{\varepsilon}_{\beta}(\cdot,t)-V^{\varepsilon}_{\beta}(\cdot,t)\right\|^2_{L^2(\mathbb{R})}\le M\le\frac{\left(1+\frac{1}{m}\right)\beta^{-2}\left\|G_W-G_V\right\|^2_{L^2(\mathbb{R})}}{1-(1+m)K_F^2T^2}.$$

Thus, we obtain

$$\left\|W^{\varepsilon}_{\beta}(\cdot,t)-V^{\varepsilon}_{\beta}(\cdot,t)\right\|_{L^2(\mathbb{R})}\le\sqrt{\frac{1+\frac{1}{m}}{1-(1+m)K_F^2T^2}}\beta^{\frac{k(t)-k(T)}{k(T)}}\left\|G_W-G_V\right\|_{L^2(\mathbb{R})}, \forall t\in[0,T]. \quad (18)$$

This completes the proof of Step 2 and also the proof of the theorem.

□

CONVERGENCE ESTIMATE

Now we are ready to state the main result

Theorem 2. *Let* $m\in\left(0,\frac{1}{K_F^2T^2}-1\right)$. *Suppose that Problem (1) has a unique solution* $u\in C([0,T];L^2(\mathbb{R}))$ *satisfying* $\left\|u(\cdot,0)\right\|_{L^2(\mathbb{R})}\le E$ *with* $E>\varepsilon$ *and the regularization parameter* $\beta=\frac{\varepsilon}{E}$ *then we have the estimate*

$$\left\| U_\beta^\varepsilon(\cdot,t) - u(\cdot,t) \right\|_{L^2(\mathbb{R})} \le 2\sqrt{\frac{1+\frac{1}{m}}{1-(1+m)K_F^2T^2}}\,\varepsilon^{\frac{k(t)}{k(T)}} E^{1-\frac{k(t)}{k(T)}}.$$

Proof. Assuming that u_β^ε is a solution of Problem (9) corresponding to the final values G, we shall estimate $\left\| u(\cdot,t) - u_\beta^\varepsilon(\cdot,t) \right\|_{L^2(\mathbb{R})}$. First we have

$$\begin{aligned} \hat{u}(\omega,t) &= \exp\Big(\psi_\alpha^\theta(\omega)(k(T)-k(t))\Big)\left(\hat{G}(\omega) - \int_t^T \exp\Big(\psi_\alpha^\theta(\omega)(k(s)-k(T))\Big)\hat{F}(\omega,s,u(\omega,s))ds\right) \\ &= \frac{\exp\Big(\psi_\alpha^\theta(\omega)(k(T)-k(t))\Big)}{1+\beta\exp\Big(|\omega|^\alpha \cos\Big(\frac{\pi\theta}{2}\Big)k(T)\Big)}\left(\hat{G}(\omega) - \int_t^T \exp\Big(\psi_\alpha^\theta(\omega)(k(s)-k(T))\Big)\hat{F}(\omega,s,u(\omega,s))ds\right) \\ &+ \frac{\beta\exp\Big(\psi_\alpha^\theta(\omega)(k(T)-k(t))\Big)\exp\Big(|\omega|^\alpha \cos\Big(\frac{\pi\theta}{2}\Big)k(T)\Big)}{1+\beta\exp\Big(|\omega|^\alpha \cos\Big(\frac{\pi\theta}{2}\Big)k(T)\Big)}\left(\hat{G}(\omega) - \int_t^T \exp\Big(\psi_\alpha^\theta(\omega)(k(s)-k(T))\Big)\hat{F}(\omega,s,u(\omega,s))ds\right). \end{aligned} \tag{19}$$

On the other hand, we get

$$\hat{u}(\omega,T) = \hat{G}(\omega) = \exp\Big(-k(T)\psi_\alpha^\theta(\omega)\Big)\left(\hat{u}(\omega,0) + \int_0^T \exp\Big(k(s)\psi_\alpha^\theta(\omega)\Big)\hat{F}(\omega,s,u(\omega,s))ds\right).$$

This implies that

$$\begin{aligned} &\hat{G}(\omega) - \int_t^T \exp\Big(\psi_\alpha^\theta(\omega)(k(s)-k(T))\Big)\hat{F}(\omega,s,u(\omega,s))ds \\ &= \exp\Big(-k(T)\psi_\alpha^\theta(\omega)\Big)\hat{u}(\omega,0) + \int_0^t \exp\Big((k(s)-k(T))\psi_\alpha^\theta(\omega)\Big)\hat{F}(\omega,s,u(\omega,s))ds. \end{aligned} \tag{20}$$

Combining (19) and (20), we obtain

$$\begin{aligned} \hat{u}(\omega,t) &= \frac{\exp\Big(\psi_\alpha^\theta(\omega)(k(T)-k(t))\Big)}{1+\beta\exp\Big(|\omega|^\alpha \cos\Big(\frac{\pi\theta}{2}\Big)k(T)\Big)}\left(\hat{G}(\omega) - \int_t^T \exp\Big(\psi_\alpha^\theta(\omega)(k(s)-k(T))\Big)\hat{F}(\omega,s,u(\omega,s))ds\right) \\ &+ \frac{\beta\exp\Big(-k(t)\psi_\alpha^\theta(\omega)\Big)\exp\Big(|\omega|^\alpha \cos\Big(\frac{\pi\theta}{2}\Big)k(T)\Big)}{1+\beta\exp\Big(|\omega|^\alpha \cos\Big(\frac{\pi\theta}{2}\Big)k(T)\Big)}\hat{u}(\omega,0) \\ &+ \frac{\beta\exp\Big(|\omega|^\alpha \cos\Big(\frac{\pi\theta}{2}\Big)k(T)\Big)}{1+\beta\exp\Big(|\omega|^\alpha \cos\Big(\frac{\pi\theta}{2}\Big)k(T)\Big)}\int_0^t \exp\Big((k(s)-k(t))\psi_\alpha^\theta(\omega)\Big)\hat{F}(\omega,s,u(\omega,s))ds. \end{aligned} \tag{21}$$

It follows from (9) and (21) that $\hat{u}(\omega,t)-u_\beta^\varepsilon(\omega,t)=B_1+B_2+B_3$,

where

$$B_1=\int_t^T \frac{\exp\left(\psi_\alpha^\theta(\omega)(k(s)-k(t))\right)}{1+\beta\exp\left(|\omega|^\alpha\cos\left(\frac{\pi\theta}{2}\right)k(T)\right)}\left[\hat{F}(\omega,s,u_\beta^\varepsilon(\omega,s))-\hat{F}(\omega,s,u(\omega,s))\right]ds,$$

$$B_2=\frac{\beta\exp\left(-k(t)\psi_\alpha^\theta(\omega)\right)\exp\left(|\omega|^\alpha\cos\left(\frac{\pi\theta}{2}\right)k(T)\right)}{1+\beta\exp\left(|\omega|^\alpha\cos\left(\frac{\pi\theta}{2}\right)k(T)\right)}\hat{u}(\omega,0),$$

$$B_3=\frac{\beta\exp\left(|\omega|^\alpha\cos\left(\frac{\pi\theta}{2}\right)k(T)\right)}{1+\beta\exp\left(|\omega|^\alpha\cos\left(\frac{\pi\theta}{2}\right)k(T)\right)}\int_0^t \exp\left((k(s)-k(t))\psi_\alpha^\theta(\omega)\right)\left[\hat{F}(\omega,s,u(\omega,s))-\hat{F}(\omega,s,u_\beta^\varepsilon(\omega,s))\right]ds.$$

This leads to

$$\left|\hat{u}(\omega,t)-u_\beta^\varepsilon(\omega,t)\right|\le|B_1|+|B_2|+|B_3|$$

$$\le\int_t^T\beta^{\frac{k(t)-k(s)}{k(T)}}\left|\hat{F}(\omega,s,u(\omega,s))-\hat{F}(\omega,s,u_\beta^\varepsilon(\omega,s))\right|ds+\beta^{\frac{k(t)}{k(T)}}|\hat{u}(\cdot,0)|+\int_0^t\beta^{\frac{k(t)-k(s)}{k(T)}}\left|\hat{F}(\omega,s,u(\omega,s))-\hat{F}(\omega,s,u_\beta^\varepsilon(\omega,s))\right|ds$$

$$\le\beta^{\frac{k(t)}{k(T)}}|\hat{u}(\cdot,0)|+\int_0^T\beta^{\frac{k(t)-k(s)}{k(T)}}\left|\hat{F}(\omega,s,u(\omega,s))-\hat{F}(\omega,s,u_\beta^\varepsilon(\omega,s))\right|ds. \tag{22}$$

Using this and (16), we conclude that

$$\left\|u(\cdot,t)-u_\beta^\varepsilon(\cdot,t)\right\|_{L^2(\mathbb{R})}^2=\left\|\hat{u}(\cdot,t)-u_\beta^\varepsilon(\cdot,t)\right\|_{L^2(\mathbb{R})}^2$$

$$\le\left(1+\frac{1}{m}\right)\beta^{\frac{2k(t)}{k(T)}}\|u(\cdot,0)\|_{L^2(\mathbb{R})}^2+(1+m)\int_{-\infty}^{+\infty}\left(\int_0^T\beta^{\frac{k(t)-k(s)}{k(T)}}\left|\hat{F}(\omega,s,u(\omega,s))-\hat{F}(\omega,s,u_\beta^\varepsilon(\omega,s))\right|ds\right)^2d\omega$$

$$\le\left(1+\frac{1}{m}\right)\beta^{\frac{2k(t)}{k(T)}}\|u(\cdot,0)\|_{L^2(\mathbb{R})}^2+(1+m)K_F^2T\beta^{\frac{2k(t)}{k(T)}}\int_0^T\beta^{\frac{-2k(s)}{k(T)}}\left\|u(\cdot,s)-u_\beta^\varepsilon(\cdot,s)\right\|_{L^2(\mathbb{R})}^2ds,$$

and thus

$$\beta^{\frac{-2k(t)}{k(T)}}\left\|u(\cdot,t)-u_\beta^\varepsilon(\cdot,t)\right\|_{L^2(\mathbb{R})}^2\le\left(1+\frac{1}{m}\right)\|u(\cdot,0)\|_{L^2(\mathbb{R})}^2+(1+m)K_F^2T\int_0^T\beta^{\frac{-2k(s)}{k(T)}}\left\|u(\cdot,s)-u_\beta^\varepsilon(\cdot,s)\right\|_{L^2(\mathbb{R})}^2ds.$$

Since $u, u_\beta^\varepsilon \in C([0,T]; L^2(\mathbb{R}))$, the function $\left\|u(\cdot,t) - u_\beta^\varepsilon(\cdot,t)\right\|_{L^2(\mathbb{R})}$ is continuous on $[0,T]$. Therefore, there exists a positive $N = \max_{t\in[0,T]} \beta^{\frac{-2k(t)}{k(T)}} \left\|u(\cdot,t) - u_\beta^\varepsilon(\cdot,t)\right\|_{L^2(\mathbb{R})}^2$. This implies that

$$N \le \left(1 + \frac{1}{m}\right) \| u(\cdot,0)\|_{L^2(\mathbb{R})}^2 + (1+m)K_F^2 T^2 N,$$

that is,

$$\beta^{\frac{-2k(t)}{k(T)}} \left\|u(\cdot,t) - u_\beta^\varepsilon(\cdot,t)\right\|_{L^2(\mathbb{R})}^2 \le N \le \frac{\left(1+\frac{1}{m}\right)\| u(\cdot,0)\|_{L^2(\mathbb{R})}^2}{1-(1+m)K_F^2 T^2}.$$

Hence, we obtain the error estimate

$$\left\|u(\cdot,t) - u_\beta^\varepsilon(\cdot,t)\right\|_{L^2(\mathbb{R})} \le E\sqrt{\frac{1+\frac{1}{m}}{1-(1+m)K_F^2T^2}}\beta^{\frac{k(t)}{k(T)}}.$$

On the other hand, using estimate (18), we get

$$\left\|u_\beta^\varepsilon(\cdot,t) - U_\beta^\varepsilon(\cdot,t)\right\|_{L^2(\mathbb{R})} \le \sqrt{\frac{1+\frac{1}{m}}{1-(1+m)K_F^2T^2}}\beta^{\frac{k(t)-k(T)}{k(T)}} \left\|G^\varepsilon - G\right\|_{L^2(\mathbb{R})} \le \sqrt{\frac{1+\frac{1}{m}}{1-(1+m)K_F^2T^2}}\beta^{\frac{k(t)-k(T)}{k(T)}} \varepsilon.$$

From the triangle inequality and these estimates, we obtain

$$\left\|U_\beta^\varepsilon(\cdot,t) - u(\cdot,t)\right\|_{L^2(\mathbb{R})} \le \left\|U_\beta^\varepsilon(\cdot,t) - u_\beta^\varepsilon(\cdot,t)\right\|_{L^2(\mathbb{R})} + \left\|u_\beta^\varepsilon(\cdot,t) - u(\cdot,t)\right\|_{L^2(\mathbb{R})}$$

$$\le \sqrt{\frac{1+\frac{1}{m}}{1-(1+m)K_F^2T^2}}\beta^{\frac{k(t)-k(T)}{k(T)}} \varepsilon + E\sqrt{\frac{1+\frac{1}{m}}{1-(1+m)K_F^2T^2}}\beta^{\frac{k(t)}{k(T)}}.$$

With $\beta = \frac{\varepsilon}{E}$, then we have the estimate

$$\left\|U_\beta^\varepsilon(\cdot,t) - u(\cdot,t)\right\|_{L^2(\mathbb{R})} \le 2\sqrt{\frac{1+\frac{1}{m}}{1-(1+m)K_F^2T^2}}\varepsilon^{\frac{k(t)}{k(T)}} E^{1-\frac{k(t)}{k(T)}}.$$

This completes the proof.

Remark 1. *If $\vartheta(t) = 1$ and $F(x,t,u) = 0$ then Problem (1) becomes a homogeneous problem. The error estimate in Theorem 2 is of order $\beta^{\frac{t}{T}}$. It is similar to the homogeneous case in [1, 6, 8].*

CONCLUSION

In this paper, we use the new regularization solution to slove a Riesz-Feller space-fractional backward diffusion problem with a time-dependent coefficient. The convergence result has been obtained under a priori bound assumptions for the exact solution and the suitable choices of the regularization parameter**.**

***Acknowledgements**: The author desires to thank the handling editor and anonymous referees for their most helpful comments on this paper.*

Chỉnh hóa cho bài toán khuếch tán ngược cấp phân số không gian Riesz-Feller với hệ số phụ thuộc thời gian

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TÓM TẮT

Trong bài báo này, chúng tôi xét một bài toán ngược cho phương trình khuếch tán cấp phân số không gian với hệ số phụ thuộc thời gian. Bài toán này có được từ phương trình khuếch tán cổ điển bằng cách thay đạo hàm bậc hai biến không gian bằng đạo hàm Riesz-Feller với $\alpha \in (0,2]$*. Đây là bài toán không chỉnh, nghĩa là nghiệm (nếu tồn tại) không phụ thuộc liên tục vào dữ liệu. Vì vậy, chúng tôi đưa ra một nghiệm chỉnh hóa mới để giải bài toán này. Sau đó, ước lượng hội tụ thu được dưới một giả định bị chặn tiên nghiệm cho nghiệm chính xác.*

***Từ khóa**: bài toán khuếch tán ngược cấp phân số không gian, bài toán không chỉnh, chỉnh hóa, ước lượng lỗi, hệ số phụ thuộc thời gian*

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